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ON THE SUBGROUPS OF AN ABELIAN GROUP

By G. A. Miller

The main object of this paper is to develop a convenient formula by means of which all the possible subgroups of a given abelian group may be directly written down. For this purpose it appears desirable to present some well known facts relating to the independent generators in a form which is especially suited to our point of view. The list of these facts is more complete than would be necessary for what follows, since it is hoped that the somewhat new form of many of them will make the list of interest and value in itself.

1. Fundamental Theorems. Any abelian group (G) is the direct product of a number of cyclic subgroups $C_1, C_2, \cdots, C_m$. The groups $C_1, C_2, \cdots, C_m$ may always be so chosen that the order of each of them is a power of a single prime number. In this case the number of these subgroups and their orders are fully determined by G provided none of them is the identity.* That is, if G is also the direct product of the cyclic groups $C_1', C_2', \cdots, C_{m'}'$, where the order of each of these groups is again a power of a single prime, then it follows that $m' = m$ and that the groups $C_1, C_2, \cdots, C_m$ and $C_1', C_2', \cdots, C_{m'}'$ can be so arranged that the order of C_α is equal to that of $C_\alpha'(\alpha = 1, 2, \cdots, m)$. In particular when the order of G is a power of a prime the orders of $C_1, C_2, \cdots, C_m$ are completely determined by G, for each must be a power of a single prime. In any case, the orders of $C_1, C_2, \cdots, C_m$ are *invariants* of G whenever each of them is a power of a prime.

When the order of each of the independent generating cyclic subgroups $C_1, C_2, \cdots, C_m$ is a power of a single prime we obtain the largest possible number of independent subgroups of G. That is, *neither G nor any of its subgroups can have more than m independent generators.* We proceed to find the smallest number of independent generators of G. The given subgroups $C_1, C_2, \cdots, C_m$ may be arranged in rows such that the orders of all of those in one row are powers of the same prime and that the order of each is equal to or greater than the order of the one which follows it in the same row. In case the

* Unless the contrary is stated it will be assumed that the order of every subgroup exceeds unity. Otherwise, the number of independent generating subgroups is indeterminate, as the identity can be added any number of times to these subgroups.

rows do not contain the same number, the vacant places may be filled by the identity. Arranging these rows in the form of a rectangle we have

$$\begin{array}{cccc} C_1, & C_2, & \cdots, & C_\alpha \\ C_{\alpha+1}, & C_{\alpha+2}, & \cdots, & C_{2\alpha} \\ . & . & & . \\ . & . & & . \\ . & . & & . \end{array}$$

By forming the products of all those in each column we obtain α *independent* cyclic subgroups such that the order of each is divisible by the orders of all those which follow it. These subgroups form the smallest possible number of generating cyclic subgroups of G. The orders of these subgroups are commonly called *the invariants* of G, since any other system of independent generating cyclic subgroups in which every order is divisible by every following order is composed of α groups of the same orders respectively. It may be observed that α is the largest number of invariants in a Sylow subgroup* of G while m is equal to the sum of the numbers of invariants of all the Sylow subgroups of G. It is clear that the independent generators of G may be so selected that their number has any arbitrary value from α to m, but this number can have no other value. Moreover, G cannot be generated by less than α cyclic subgroups even if these subgroups are not independent.

Whenever the independent generators of G are so chosen that each of them is divisible by all those which follow it their number must be α, and when the order of each is a power of a single prime their number must be m. These two numbers are equal only when the order of G is a power of a single prime. Since the former method leads to the smallest number of invariants it seems appropriate to call the orders of these independent generators *the invariants* of G, although the latter method has some advantages. The choice of invariants such that their number lies between α and m seems less natural. We evidently arrive at the α invariants if we choose the independent generators in the ordinary way; *i.e.* start with an operator of highest order and then select any other operator such that the two generate the largest possible subgroup. The orders of two independent generators of this subgroup are the first two invariants of G. If we add to this subgroup another operator so that the three generate the largest possible subgroup, we arrive at the third invariant, etc.

* If the order of a group is divisible by p^α (p being a prime), but not by $p^\alpha+1$, its subgroups of order p^α are called Sylow subgroups.

A marked difference between the two given methods of arriving at the invariants of G should perhaps be emphasized; viz., the orders of the independent generators of G are completely determined by m, but not by a. That is, if two sets of m independent generating cyclic subgroups of G were given the orders of the subgroups of one set would be the same as those of the other; but if two sets of a independent generating cyclic subgroups of G were given the orders of those of one set could generally vary a great deal from the orders of those of the other. The necessary and sufficient condition that the orders of these two sets be the same is that each of the Sylow subgroups of G contain the same number of invariants.

If G is the direct product of a series of subgroups $G_1, G_2, \cdots, G_\lambda$ we may select a set of independent generators of G by taking any set of independent generators of each one of these subgroups. Suppose that $G_1, G_2, \cdots, G_\lambda$ are the Sylow subgroups of G. Any operator (t) of G will have a constituent, which may be the identity, from each one of these subgroups and the order of t will be the product of the orders of these constituents. In order to determine the number of the operators of a given order in G it is only necessary to determine the number of operators of a given order in each of the Sylow subgroups. That is, if the order of t is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$ ($p_1, p_2, \cdots, p_\lambda$ being prime numbers), the number of operators of G whose order is equal to the order of t is the product of the numbers of operators of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_\lambda^{\alpha_\lambda}$ in the respective Sylow subgroups of G. We proceed to determine this number.

2. Number of operators of a given order in a group of order p^m. Let G be any group of order p^m whose invariants are $p^{\alpha_1}, p^{\alpha_2}, \cdots, p^{\alpha_\gamma}$ ($\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_\gamma > 0$). Let $m_1 = \gamma$ represent the number of invariants $\geqq p$, m_2 the number of those $\geqq p^2$, $\cdots$, m_{α_1} the number of those $= p^{\alpha_1}$. To determine the number of the operators of order p^β ($1 \leqq \beta \leqq \alpha_1$) it is only necessary to find the order of the group generated by all the operators whose orders divide p^β and subtract from this number the order of the group formed by all the operators whose orders divide $p^{\beta-1}$. That is, the number of operators of order p^β in G is equal to *

$$p^{m_1+m_2+\cdots+m_\beta} - p^{m_1+m_2+\cdots+m_{\beta-1}} = p^{m_1+m_2+\cdots+m_{\beta-1}}(p^{m_\beta} - 1).$$

* Heffter, *Crelle*, vol. 119(1898), p. 261; Netto, *Vorlesungen über Algebra*, vol. 2 (1900) p. 248.

This result may also be obtained by observing that all the operators whose orders divide p^β constitute a group which has m invariants; m_β of these are equal to p^β while the remaining $m - m_\beta$ are the invariants of G which are less than p^β. If $a_\rho \geqq \beta > a_{\rho+1}$ the order of this group is p^M, where $M = \beta m_\rho + a_{\rho+1} + \cdots + a_\gamma$. The order of the group which is composed of all the operators whose orders divide $p^{\beta-1}$ is found in the same way and the difference of these orders is the number of operators of order p^β which are contained in G.

3. Number of different types of subgroups in a group of order p^m. Let G have the same invariants as in the preceding paragraphs. It is well known that any ρ powers of p, such as $p^{\beta_1}, p^{\beta_2}, \cdots, p^{\beta_\rho}$, are the invariants of some subgroup of G, provided $\beta_1 \geqq a_1, \beta_2 \geqq a_2, \cdots, \beta_\rho \geqq a_\rho$*. As these conditions are necessary as well as sufficient, $\rho \geqq \gamma$. Hence the number of different types of subgroups having just ρ invariants ($\rho > 1$) is given by the formula

$$\sum_{\lambda_\rho = a_\rho - 1}^{\lambda_\rho = 0} \cdots \sum_{\lambda_3 = a_3 - 1}^{\lambda_3 = \lambda_4} \sum_{\lambda_2 = a_2 - 1}^{\lambda_2 = \lambda_3} a_1 - \lambda_2.$$

There are evidently just $a_1 + 1$ cyclic subgroups†. For instance, the group whose invariants are p^5, p^3, p^2, p^2 contains 9 cyclic subgroups, 24 which have two invariants, 39 which have three invariants, and 54 which have four invariants. From the fact that $\lambda_\rho = 0$ whenever $a_\rho = 1$, it follows that *the number of the different types of subgroups which have ρ invariants is equal to the number which have $\rho - 1$ invariants whenever $a_\rho = 1$.*

The number of the different types of subgroups which have just ρ invariants may be directly obtained, without the use of a formula, as follows: First suppose that all the invariants after the first are equal to p. There are clearly a_1 such subgroups. Then suppose that the second invariant is p^2 while all those which follow it are equal to p. As we may suppose that the invariants are always arranged in order of magnitude, there are $a_1 - 1$ such subgroups. Continuing to multiply the second invariant by p until it is equal to p^{a_2}, we find the number of different types of subgroups in which all the invariants after the first two are equal to unity. This number is clearly equal to $a_1 + (a_1 - 1) + \cdots + (a_1 - a_2 + 1)$. We then increase the third invariant successively by a multiple of p, while the first and second invariant can have all possible values such that the first $\geqq$ the second $\geqq$ the third. Continuing this process

* Burnside, *Theory of groups of finite order*, 1897, p. 58.

† The identity is here included among the cyclic subgroups of G.

until we arrive at ρ invariants we obtain all the possible types of subgroups which have just ρ invariants.

When the order of G is not a power of a prime number the number of its different types of subgroups is the product of the number of the different types of subgroups in its Sylow subgroups. The identity and the entire group are included under the term subgroup in this statement. Hence the given formula can be directly employed to determine the number of different types of subgroups in any abelian group.

4. Total number of subgroups in a group of order p^m. It will only be necessary to determine the number of the subgroups whose invariants are $p^{\beta_1}, p^{\beta_2}, \cdots, p^{\beta_\rho}$ $(\beta_1 \geqq a_1, \beta_2 \geqq a_2, \cdots, \beta_\rho \geqq a_\rho)$.* Let $s_1, s_2, \cdots, s_\rho$ represent the independent generators of such a subgroup, the orders of these independent generators being $p^{\beta_1}, p^{\beta_2}, \cdots, p^{\beta_\rho}$ respectively. For s_1 we may choose any one of the $(p^{m_{\beta_1}} - 1)p^{m_1 + m_2 + \cdots + m_{\beta_1} - 1}$ operators of G whose order is p^{β_1}. We proceed to determine the number of ways in which s_2 can be chosen. It is clear that we may take any operator of order p^{β_2} except those which generate the same subgroup of order p that s_1 does. As the number of the latter is equal to $p - 1$ times the order of the group composed of all the operators in G whose orders divide $p^{\beta_2 - 1}$, the number of ways in which s_2 may be chosen is

$$(p^{m_{\beta_2}} - 1)p^{m_1 + m_2 + \cdots + m_{\beta_2} - 1} - (p - 1)p^{m_1 + m_2 + \cdots + m_{\beta_2} - 1},$$

or

$$(p^{m_{\beta_2}} - p)p^{m_1 + m_2 + \cdots + m_{\beta_2} - 1}.$$

Similarly, we may take any operator of order p^{β_3} for s_3 with the exception of those which generate a group of order p contained in $\{s_1, s_2\}$. The number of the latter is equal to $p^2 - 1$ times the order of the group composed of all the operators of G whose orders divide $p^{\beta_3 - 1}$. Hence the third independent generator can be chosen in

$$(p^{m_{\beta_3}} - p^2)p^{m_1 + m_2 + \cdots + m_{\beta_3} - 1}$$

ways. The number of ways in which s_ρ can be chosen is

$$(p^{m_{\beta_\rho}} - p^{\rho - 1})p^{m_1 + m_2 + \cdots + m_{\beta_\rho} - 1}.$$

The product of all these numbers gives the number of ways in which the

* The independent generators and the invariants are always supposed to be arranged in order of magnitude, the largest coming first.

independent generators of such a subgroup may be chosen from the operators of G. If we divide this product by the number of ways in which these independent generators can be chosen within the subgroup we obtain a quotient which represents the number of the subgroups of G which are of type $(\beta_1, \beta_2, \cdots, \beta_\rho)$.

If we put $A = \sum_{x=0}^{x=\rho-1} (\rho - x)(m_{\beta_\rho - x + 1} + m_{1+\beta_\rho - x + 1} + \cdots + m_{-1+\beta_{\rho-x}})$, this quotient is

$$\frac{(p^{m\beta_1} - 1)(p^{m\beta_2} - p) \cdots (p^{m\beta_\rho} - p^{\rho-1})\, p^A}{\text{Same expression except that } m \text{ is replaced by } n},$$

where n_a represents the number of the invariants which are $\leqq p^a$ in the required subgroups, $m_{\beta\rho+1} = m_0 = n_{\beta_\rho+1} = n_0 = 0$, and $m_{\beta_\rho - x+1} + \cdots + m_{-1+\beta_\rho - x} = 0 = (n_{\beta_\rho - x+1} + \cdots + n_{-1+\beta_\rho - x})$ whenever $\beta_{\rho-x+1} = \beta_{\rho-x}$.

By means of this formula we can determine the total number of subgroups of a given type in G. A preceding formula gives the number of possible types. Hence the two formulas together enable us to determine all the possible subgroups of a group of order p^m. When the order of G is not a power of a prime number the number of its subgroups is the product of the numbers of the subgroups in the Sylow subgroups of G. Hence the given formulas may be employed to determine the number of subgroups in any abelian group.

STANFORD UNIVERSITY,
FEBRUARY 1904.

NOTE ON THE CONTINUED PRODUCT OF THE OPERATORS OF ANY GROUP OF FINITE ORDER

By William B. Fite

Miller has shown* that if an abelian group contains more than one operator of order two, the continued product of all its operators is the identity; and that if it contains only one such operator, this operator is the continued product of all the operators. In this note I show by the most elementary considerations that non-abelian groups of finite order possess an analogous property. So far as I am aware no one has ever called attention to this fact.

Let G be a non-abelian group of odd order. We can arrange its operators in such a way that every operator is followed by its inverse, and the continued product in this order is the identity. Hence the continued product of these operators in any order is an operator of the commutator subgroup. Moreover, the operators can be arranged so that the product is any given commutator t. For, if $A_i^{-1} A_j A_i = A_j t$, then

$$A_i^{-1} A_i A_j A_j^{-1} \cdots A_n = 1,$$

and

$$A_j^{-1} \cdots A_n A_i^{-1} A_i A_j = 1.$$

Consequently

$$A_j^{-1} \cdots A_n A_i^{-1} A_j A_i = t.$$

Suppose now that G is of even order and that K, of order k, is its commutator subgroup. Since the quotient group G/K is abelian, the continued product of its operators is the identity or an operator of order two. If now we arrange the operators of G by taking first one operator from each division of G/K in turn, then another operator from each of these divisions in the same order, and so on until we have all the operators of G, the continued product of these operators in this order corresponds to the k^{th} power of the product of the operators of G/K. This latter is the identity unless k is odd and G/K has only one operator of order two. In the exceptional case, the product of the operators of G taken in a certain order corresponds to the operator of order two in the quotient group G/K. If we take the operators in any other

* Annals of Mathematics, ser. 2, vol. 4 (1903), p. 189. See also *Giornali di Matematiche di Battaglini*, vol. 41 (1903), pp. 203, 336.

order, the only effect on the product is to multiply it by some operator of the commutator subgroup. Hence:—

The continued product in any order of the operators of a group G whose commutator subgroup is K, of order k, is an operator of K, except when k is odd and the quotient group G/K has only one operator of order two; in this exceptional case the continued product of the operators of G corresponds to the operator of order two of G/K.

In particular, if G is of class l and order $2^a m$, where m is odd, the product of its operators in any order is an operator of the commutator subgroup, except when the subgroup of order 2^a is cyclic.

CORNELL UNIVERSITY,
MAY, 1904.

REDUCTION OF AN ELLIPTIC INTEGRAL TO LEGENDRE'S NORMAL FORM

By Norman R. Wilson

In his *Traité des fonctions elliptiques*, Legendre has shown that the integral $\int \frac{dx}{\sqrt{\psi(x)}}$ (x real and $\psi(x)$ a quartic in x with real coefficients) can be reduced to the normal form $\int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$ (κ real and less than 1) by a real transformation of the variables. The problem is considered under six heads. Cayley observes (*Elliptic Functions* § 414) that of these only five are distinct. The following note reduces the number to three, an invariantic method being used.* It is assumed that the roots of $\psi(x) = 0$ are all distinct, and further that x is restricted to an interval such that the expression under the radical is positive.

Writing the integral in the homogeneous form, it becomes

$$\int \frac{x_2dx_1 - x_1dx_2}{\sqrt{A_0x_1^4 + 4A_1x_1^3x_2 + 6A_2x_1^2x_2^2 + 4A_3x_1x_2^3 + A_4x_2^4}}.$$

Resolve the quartic in the denominator into the real quadratic factors, $f \equiv a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2$ and $f' \equiv b_0x_1^2 + 2b_1x_1x_2 + b_2x_2^2$, and let the roots of $f = 0$ and $f' = 0$ be α_1,α_2 and β_1,β_2 respectively. If only two of these are real, they are to be β_1 and β_2, and if all are real, β_1 and β_2 are to be the two between† which x lies.

1. The sign of $\psi(x)$ and of the resultant of f and f'. If R denote the resultant, and the quartic have all its roots finite,

$$R = a_0^2b_0^2(\alpha_1 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1)(\alpha_2 - \beta_2).$$

(Weber, *Algebra* §53).

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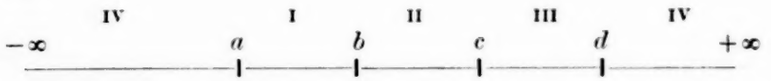
* The reduction can also be made, in a similar manner, by very elementary algebra.

† The word *"between"* is used in the sense of projective geometry. Thus if *abcd* be the roots in ascending order of magnitude, x would be said to lie between d and a if $x > d$ or $x < a$.

(*a*) *If* $\psi(x) = 0$ *has all its roots real and finite.* Since β_1, β_2 are the roots between which x lies, neither or both of these lie between a_1 and a_2. R will therefore be positive. Representing the values of the roots geometrically as a, b, c, d, and remembering that $\psi(x) \equiv A_0(x - a_1)(x - a_2)(x - \beta_1)(x - \beta_2)$,



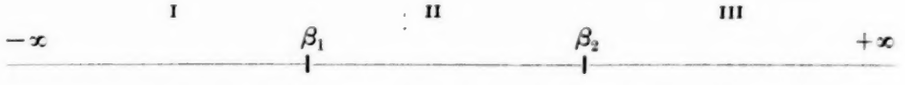
it is evident that when A_0 is positive, in order that $\psi(x)$ shall be positive, x must lie in the interval II or IV, and when A_0 is negative, in the interval I or III. Remembering that in each case β_1, β_2 are the extremities of these intervals, it is easily seen from the figures that the sign of R is positive.

(*b*) *If* $\psi(x) = 0$ *has complex, but no infinite roots.* When there are two real roots, β_1, β_2, $a_1 - \beta_1$, $a_2 - \beta_1$ and $a_1 - \beta_2$, $a_2 - \beta_2$ will be pairs of conjugate complexes; and when all the roots are complex $a_1 - \beta_1$, $a_2 - \beta_2$ and $a_1 - \beta_2$, $a_2 - \beta_1$ will be conjugate. Hence in both cases R will be positive.

(*c*) $\psi(x) = 0$ *has an infinite root.* In this case $A_0 = 0$. If the integral is to remain elliptic, a_0 and b_0 cannot both vanish. If $a_0 = 0$, all the roots of $\psi(x) = 0$ are real. Since x is to lie between β_1 and β_2, the finite root of $f = 0$ cannot lie between them. Writing f in the form $a(x - a)$ and f' as $b(x - \beta_1)(x - \beta_2)$, and substituting for the coefficients in the expression for R (Weber, *Algebra*, § 53), we find

$$R = a^2b^2(a - \beta_1)(a - \beta_2),$$

which is positive since a must lie outside the interval $\beta_1\,\beta_2$.

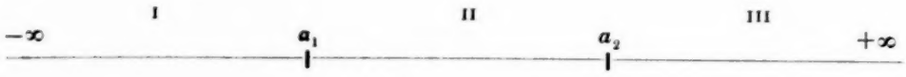


Now $\psi(x) \equiv ab(x - a)(x - \beta_1)(x - \beta_2)$. If $\psi(x)$ is positive, ab and $x - a$ must have different signs. Representing the values of the roots β_1, β_2 geometrically, x lies in the interval II. If ab be positive, a lies in III and if ab be negative, in I. In both cases R is positive.

On the other hand, if $b_0 = 0$, and we write $f \equiv a(x - a_1)(x - a_2)$ and $f \equiv b(x - \beta)$, then

$$R = a^2b^2(a_1 - \beta)(a_2 - \beta).$$

If a_1, a_2 are complex, this is positive. If they are real, β cannot lie between them. In the latter case, $x - a_1$ and $x - a_2$ have the same sign. Hence $(x - \beta)$ must have the same sign as ab.



If ab be positive, β and x must both lie in III, $\beta < x$; if ab be negative, β and x must both lie in I and $x < \beta$.

2. The Algebraic Reduction. The values of λ for which $f + \lambda f'$ is a perfect square are given by $(a_0 + \lambda b_0)(a_2 + \lambda b_2) - (a_1 + \lambda b_1)^2 = 0$, or

$$D''\lambda^2 + 2D'\lambda + D = 0 \quad \ldots\ldots\ldots\ldots \quad (\alpha),$$

where $D \equiv 2(a_0 a_2 - a_1^2)$; $D'' = 2(b_0 b_2 - b_1^2)$; $D' = a_0 b_2 + a_2 b_0 - 2a_1 b_1$.

Now $D'^2 - DD''$ is easily seen to be equal to R, which we have shown to be positive (§1). Hence the roots λ_1 and λ_2 of (α) are real and unequal. The coefficient of x_1^2 in $f + \lambda f'$ being $(a_0 + \lambda b_0)$ we may write

$$\left.\begin{aligned} f + \lambda_1 f' &\equiv (a_0 + \lambda_1 b_0)(x_1 - \eta x_2)^2 \equiv \epsilon_1 y_1^2, \\ f + \lambda_2 f' &\equiv (a_0 + \lambda_2 b_0)(x_1 - \theta x_2)^2 \equiv \epsilon_2 y_2^2, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\beta)$$

where ϵ_1, ϵ_2 stand for ± 1 according as $(a_0 + \lambda_1 b_0)$ and $(a_0 + \lambda_2 b_0)$ are respectively positive and negative. Hence x_1, x_2 and y_1, y_2 are connected by the linear substitution

$$\left.\begin{aligned} y_1 &= \pm \sqrt{\frac{a_0 + \lambda_1 b_0}{\epsilon_1}}\ (x_1 - \eta x_2), \\ y_2 &= \pm \sqrt{\frac{a_0 + \lambda_2 b_0}{\epsilon_2}}\ (x_1 - \theta x_2), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\gamma)$$

the signs of the square roots being still at our disposal. Solving (β) for f and f', the quartic ff'' reduces to

$$\frac{-\lambda_1}{(\lambda_1 - \lambda_2)^2}\left(y_2^2 - \frac{\epsilon_1}{\epsilon_2} y_1^2\right)\left(y_2^2 - \frac{\epsilon_1}{\epsilon_2} \cdot \frac{\lambda_2}{\lambda_1} \cdot y_1^2\right).$$

Writing μ and ν for $\sqrt{(a_0 + \lambda_1 b_0)\epsilon_1}$ and $\sqrt{(a_0 + \lambda_2 b_0)\epsilon_2}$, we have

$$\mu\nu = \sqrt{\epsilon_1\epsilon_2(a_0 + \lambda_1 b_0)(a_0 + \lambda_2 b_0)} = \sqrt{\frac{-2\epsilon_1\epsilon_2}{D''}}\,(a_0 b_1 - a_1 b_0),$$

on substituting for $(\lambda_1 + \lambda_2)$ and $\lambda_1\lambda_2$ from (α), and performing simple reductions. The equation giving η and θ (the Jacobian; *cf.* Salmon, *Algebra*, §177) is

$$(a_0 b_1 - a_1 b_0)x_1^2 + (a_0 b_2 - a_2 b_0)x_1 x_2 + (a_1 b_2 - a_2 b_1)x_2^2 = 0,$$

whence

$$\begin{aligned}(a_0 b_1 - a_1 b_0)^2(\eta - \theta)^2 &= (a_2 b_0 - a_2 b_0)^2 - 4(a_0 b_1 - a_1 b_0)(a_1 b_2 - a_2 b_1)\\ &= R.\end{aligned}$$

Now R has been shown to be positive in §1. Hence θ and η are real, and, therefore, the substitution (γ) is real. Further, $\mu\nu(\eta - \theta)$ reduces to $\sqrt{\frac{-2\epsilon_1\epsilon_2 R}{D''}}$. We choose the signs of μ and ν so that $\mu\nu(\eta - \theta)$ is equal to the positive value of this square root. Now

$$\begin{vmatrix} y_2 & dy_2 \\ y_1 & dy_1 \end{vmatrix} = \begin{vmatrix} \nu x_1 - \nu\theta x_2, & \nu dx_1 - \nu\theta dx_2 \\ \mu x_1 - \mu\eta x_2, & \mu dx_1 - \mu\eta dx_2 \end{vmatrix} = \begin{vmatrix} \nu, & -\nu\theta \\ \mu, & -\mu\eta \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 \\ dx_1 & dx_2 \end{vmatrix}.$$

Hence $(x_2 dx_1 - x_1 dx_2) = (y_2 dy_1 - y_1 dy_2) \div \mu\nu(\eta - \theta)$. Further, from (α), we have

$$(\lambda_1 - \lambda_2)^2 = \frac{4(D'^2 - DD'')}{D''^2} = \frac{4R}{D''^2}.$$

Substituting in our integral, and returning to non-homogeneous variables, we have

$$\sqrt{\frac{2}{\lambda_1 \cdot D'' \cdot \epsilon_1\epsilon_2}} \int \frac{dy}{\sqrt{\left(1 - \frac{\epsilon_1}{\epsilon_2}y^2\right)\left(1 - \frac{\epsilon_1}{\epsilon_2}\frac{\lambda_2}{\lambda_1}y^2\right)}}.$$

Also, since the modulus, $\mu\nu(\theta - \eta)$ or $\sqrt{\frac{-2\epsilon_1\epsilon_2 R}{D''}}$, of the substitution (γ) must be real, $\epsilon_1\epsilon_2$ must have a sign opposite to that of D''. We consider the following cases:

(*a*) $\psi(x) = 0$ *has four real roots.* When the roots are also finite, D' must be a function of the differences, easily seen to be

$$\tfrac{1}{2} A_0 \{(a_1 - \beta_1)(a_2 - \beta_2) + (a_1 - \beta_2)(a_2 - \beta_1)\}.$$

If A_0 be positive, a reference to the geometrical representation of § 1(*a*) will show that both terms within the bracket are negative. If A_0 be negative, a similar result follows. If $A_0 = 0$ and $b_0 = 0$ the expression for D' becomes $ab\{(a_1 - \beta) + (a_2 - \beta)\}$, which a reference to § 1(*c*) shows to be negative. If $a_0 = 0$, D' becomes $ab\{(\beta_1 - a) + (\beta_2 - a)\}$, which is again negative by § 1(*c*).

In all cases, therefore, D' is negative. Since $f = 0$ and $f' = 0$ have real roots, D and D'' are both negative. Hence both roots λ_1, λ_2 of (*a*) are negative. Let $|\lambda_1|$ be the greater. Then we may write $\lambda^2 = \frac{\lambda_2}{\lambda_1}$ (λ real and $|\lambda|$ less than 1). Since D'' is negative, $\epsilon_1\epsilon_2 = -1$, and our integral reduces to

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}}, \qquad 0 < \lambda^2 < 1.$$

(*b*) $\psi(x) = 0$ *has two real and two complex roots.* In this case, D is positive and D'' negative. Hence (*a*) has a positive and a negative root, λ_2 and λ_1 respectively. Further $\epsilon_1\epsilon_2 = -1$ since D'' is negative. Writing $\frac{\lambda_2}{\lambda_1} = -\lambda^2$, we obtain our integral in the form

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 - y^2)(1 + \lambda^2 y^2)}}.$$

(*c*) $\psi(x) = 0$ *has four complex roots.* If $\psi(x) = 0$ is to be positive, A_0 must be positive. Hence D' or $A_0\{(a_1 - \beta_1)(a_2 - \beta_2) + (a_2 - \beta_1)(a_1 - \beta_2)\}$, being the sum of two products of conjugate complex quantities, is positive. So are D and D''. Hence the two roots of (*a*) are negative. Choose $|\lambda_1| > |\lambda_2|$; since the roots are unequal, this may be done. Writing $\frac{\lambda_2}{\lambda_1} = \lambda^2$, we obtain

$$\sqrt{\frac{2}{-\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 + y^2)(1 + \lambda^2 y^2)}}, \qquad 0 < \lambda^2 < 1.$$

Hence we have reduced the integral to the form:

$$\text{real constant} \times \int \frac{dy}{\sqrt{(1 \pm y^2)(1 \pm \lambda^2 y^2)}},$$

Writing μ and ν for $\sqrt{(a_0 + \lambda_1 b_0)\epsilon_1}$ and $\sqrt{(a_0 + \lambda_2 b_0)\epsilon_2}$, we have

$$\mu\nu = \sqrt{\epsilon_1\epsilon_2(a_0 + \lambda_1 b_0)(a_0 + \lambda_2 b_0)} = \sqrt{\frac{-2\epsilon_1\epsilon_2}{D''}}\,(a_0 b_1 - a_1 b_0),$$

on substituting for $(\lambda_1 + \lambda_2)$ and $\lambda_1\lambda_2$ from (α), and performing simple reductions. The equation giving η and θ (the Jacobian; *cf.* Salmon, ***Algebra***, §177) is

$$(a_0 b_1 - a_1 b_0)x_1^2 + (a_0 b_2 - a_2 b_0)x_1 x_2 + (a_1 b_2 - a_2 b_1)x_2^2 = 0,$$

whence

$$\begin{aligned}(a_0 b_1 - a_1 b_0)^2(\eta - \theta)^2 &= (a_2 b_0 - a_2 b_0)^2 - 4(a_0 b_1 - a_1 b_0)(a_1 b_2 - a_2 b_1)\\ &= R.\end{aligned}$$

Now R has been shown to be positive in §1. Hence θ and η are real, and, therefore, the substitution (γ) is real. Further, $\mu\nu(\eta - \theta)$ reduces to $\sqrt{\frac{-2\epsilon_1\epsilon_2 R}{D''}}$. We choose the signs of μ and ν so that $\mu\nu(\eta - \theta)$ is equal to the positive value of this square root. Now

$$\begin{vmatrix} y_2 & dy_2 \\ y_1 & dy_1 \end{vmatrix} = \begin{vmatrix} \nu x_1 - \nu\theta x_2, & \nu dx_1 - \nu\theta dx_2 \\ \mu x_1 - \mu\eta x_2, & \mu dx_1 - \mu\eta dx_2 \end{vmatrix} = \begin{vmatrix} \nu, & -\nu\theta \\ \mu, & -\mu\eta \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_2 \\ dx_1 & dx_2 \end{vmatrix}.$$

Hence $(x_2\,dx_1 - x_1\,dx_2) = (y_2\,dy_1 - y_1\,dy_2) \div \mu\nu(\eta - \theta)$. Further, from (α), we have

$$(\lambda_1 - \lambda_2)^2 = \frac{4(D'^2 - DD'')}{D''^2} = \frac{4R}{D''^2}.$$

Substituting in our integral, and returning to non-homogeneous variables, we have

$$\sqrt{\frac{2}{\lambda_1 \cdot D'' \cdot \epsilon_1\epsilon_2}} \int \frac{dy}{\sqrt{\left(1 - \frac{\epsilon_1}{\epsilon_2}y^2\right)\left(1 - \frac{\epsilon_1}{\epsilon_2}\frac{\lambda_2}{\lambda_1}y^2\right)}}.$$

Also, since the modulus, $\mu\nu(\theta - \eta)$ or $\sqrt{\frac{-2\epsilon_1\epsilon_2 R}{D''}}$, of the substitution (γ) must be real, $\epsilon_1\epsilon_2$ must have a sign opposite to that of D''. We consider the following cases:

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In all cases, therefore, D' is negative. Since $f = 0$ and $f' = 0$ have real roots, D and D'' are both negative. Hence both roots λ_1, λ_2 of (*a*) are negative. Let $|\lambda_1|$ be the greater. Then we may write $\lambda^2 = \dfrac{\lambda_2}{\lambda_1}$ (λ real and $|\lambda|$ less than 1). Since D'' is negative, $\epsilon_1\epsilon_2 = -1$, and our integral reduces to

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}}, \qquad 0 < \lambda^2 < 1.$$

(*b*) $\psi(x) = 0$ *has two real and two complex roots.* In this case, D is positive and D'' negative. Hence (*a*) has a positive and a negative root, λ_2 and λ_1 respectively. Further $\epsilon_1\epsilon_2 = -1$ since D'' is negative. Writing $\dfrac{\lambda_2}{\lambda_1} = -\lambda^2$, we obtain our integral in the form

$$\sqrt{\frac{2}{\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 - y^2)(1 + \lambda^2 y^2)}}.$$

(*c*) $\psi(x) = 0$ *has four complex roots.* If $\psi(x) = 0$ is to be positive, A_0 must be positive. Hence D' or $A_0\{(a_1 - \beta_1)(a_2 - \beta_2) + (a_2 - \beta_1)(a_1 - \beta_2)\}$, being the sum of two products of conjugate complex quantities, is positive. So are D and D''. Hence the two roots of (*a*) are negative. Choose $|\lambda_1| > |\lambda_2|$; since the roots are unequal, this may be done. Writing $\dfrac{\lambda_2}{\lambda_1} = \lambda^2$, we obtain

$$\sqrt{\frac{2}{-\lambda_1 D''}} \int \frac{dy}{\sqrt{(1 + y^2)(1 + \lambda^2 y^2)}}, \qquad 0 < \lambda^2 < 1.$$

Hence we have reduced the integral to the form:

$$\text{real constant} \times \int \frac{dy}{\sqrt{(1 \pm y^2)(1 \pm \lambda^2 y^2)}},$$

where λ is real, and in the first and last cases $\lambda^2 < 1$.

3. The Trigonometric Reduction.

(*a*) $\psi(x) = 0$ *has four real roots.* f' has been transformed into $(1 - y^2)$ multiplied by a certain constant (see equations (β)). Hence the anharmonic ratio $(x, \beta_1, \alpha_1, \beta_2)$ becomes $(y, 1, \pm 1/\lambda, -1)$. The transforming substitution is

$$\frac{(x-\beta_1)}{(x-\beta_2)} \cdot \frac{(\alpha_1-\beta_2)}{(\alpha_1-\beta_2)} = \frac{(y-1)(1\pm\lambda)}{(y+1)(1\pm\lambda)} .$$

But $|\lambda| < 1$ (see § 1(*a*)). Referring to § 1(*a*), if $\beta_1 < \alpha_1 < \alpha_2 < \beta_2$, x must lie in the interval IV, and $\frac{x-\beta_1}{x-\beta_2}$ is positive, while $\frac{\alpha_1-\beta_1}{\alpha_1-\beta_2}$ is negative. If $\alpha_1 < \beta_1 < \beta_2 < \alpha_2$ x lies in the interval II, and $\frac{x-\beta_1}{x-\beta_2}$ is negative, while $\frac{\alpha_1-\beta_1}{\alpha_1-\beta_2}$ is positive. In both cases $\frac{y-1}{y+1}$ must be negative. If one of the roots of $\psi = 0$ is infinite, and it be a root of $f = 0$, then taking α_1 as the other root, the above argument is not affected. If it be a root of $f' = 0$, β_1 say, the anharmonic equality reduces to

$$\frac{\alpha_1-\beta_1}{\alpha_1-\beta_2} = \frac{(y-1)(1\pm\lambda)}{(y+1)(1\pm\lambda)} .$$

Now α_1 and x must be on different sides of β_2 (see § 1 (*c*)). Hence $\frac{\alpha_1-\beta}{x-\beta_2}$ is negative, and therefore the same is true of $\frac{y-1}{y+1}$. Hence in all cases $|y| < 1$. Write $\sin\varphi$ for y and κ for λ and we obtain

$$\sqrt{\frac{2}{\lambda_1 D'}} \int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}} ,$$

or, if it is desired to express it in terms of more familiar invariants,

$$\sqrt{\frac{2}{\sqrt{R_1 - D'}}} \int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}} .$$

(*b*) $\psi(x) = 0$ *has two real and two complex roots.* Since $\sqrt{\frac{2}{\lambda_1 D'}}$ is real,

and our integral real, $(1 - y^2)$ must be positive, and therefore $|y| < 1$. Put $\cos\varphi$ for y and we get

$$-\sqrt{\frac{2}{\lambda_1 D''}}\int\frac{d\varphi}{\sqrt{1+\lambda^2\cos^2\varphi}} = -\sqrt{\frac{2}{\lambda_1 D''(1+\lambda^2)}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},$$

where $\kappa^2 = \frac{\lambda^2}{1+\lambda^2}$, and is therefore positive and less than 1. Since $\lambda^2 = -\frac{\lambda_2}{\lambda_1}$, this becomes

$$-\sqrt{\frac{2}{(\lambda_1-\lambda_2)D''}} = -\frac{1}{\sqrt{R}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},$$

where $\kappa^2 = \frac{\lambda_2}{\lambda_2-\lambda_1}$ From the signs of D, D'', it is easily seen that $\lambda_1 = \frac{-D'+\sqrt{R}}{D''}$, and hence $\kappa^2 = \frac{D'+\sqrt{R}}{2\sqrt{R}}$.

(c) $\psi(x) = 0$ *has four complex roots.* Put $y = \tan\varphi$, and our integral becomes

$$\sqrt{\frac{-2}{\lambda_1 D''}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}},$$

where $\kappa^2 = 1 - \lambda^2$, and is therefore positive and less than 1 (see §2(c)). Hence, substituting for λ^2, we have $\kappa^2 = \frac{\lambda_1-\lambda_2}{\lambda_1}$, and by (a) we find $\lambda_1 = \frac{-D-\sqrt{R}}{D''}$ (see §2 (c)). Hence we obtain

$$\sqrt{\frac{2}{D'+\sqrt{R}}}\int\frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}.$$

We may summarize as follows: To reduce to Legendre's normal form the integr al $\int\frac{dx}{\sqrt{\psi(x)}}$ (x real and $\psi(x)$ a quartic in x with real coefficients and unequal roots), resolve the quartic into real quadratic factors, f and f', where, if only two of the roots of $\psi(x) = 0$ are real, $f' = 0$ contains them, and if they are all real, $f' = 0$ contains the pair between which x lies. Let $f \equiv a_0x^2 + 2a_1x + a_2$, and $f' \equiv b_0x^2 + 2b_1x + b_2$. Find the invariants

$$D \equiv 2(a_0a_2 - a_1^2),\ D'' \equiv 2(b_0b_2 - b_1^2),\ D' = a_0b_2 + a_2b_0 - 2a_1b_1,$$
$$R \equiv (a_0b_2 - a_2b_0)^2 - 4(a_0b_1 - a_1b_0)(a_1b_2 - a_2b_1).$$

Find also the roots, necessarily real and unequal, of the quadratic equation

$$D''\lambda^2 + 2D'\lambda + D = 0.$$

Call the lesser of these λ_1 and the greater λ_2. Then the normal form, and the transforming substitution are given in the following table, where in every case $0 < \kappa^2 < 1$.

CASE.	NORMAL FORM.	κ^2	κ^2	TRANSFORMATION.
Four real roots or $D-$; $D''-$	$\sqrt{\frac{2}{\sqrt{R}-D'}}\int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$	$\frac{\lambda_2}{\lambda_1}$	$\frac{R+D'^2+2D'\sqrt{R}}{DD''}$	$\pm\sqrt{\frac{f+\lambda_1 f'}{f+\lambda_2 f'}} = \sin\varphi$
Two real roots, two complex roots, or $D+$; $D''-$	$\frac{-1}{\sqrt[4]{R}}\int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$	$\frac{\lambda_2}{\lambda_2-\lambda_1}$	$\frac{D'+\sqrt{R}}{2\sqrt{R}}$	$\pm\sqrt{\frac{f+\lambda_1 f'}{f+\lambda_2 f'}} = \cos\varphi$
Four complex roots or $D+$; $D''+$	$\sqrt{\frac{2}{D'+\sqrt{R}}}\int \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}$	$\frac{\lambda_1-\lambda_2}{\lambda_1}$	$\frac{2\sqrt{R}}{\sqrt{R}+D'}$	$\pm\sqrt{\frac{f+\lambda_1 f'}{f+\lambda_2 f'}} = \tan\varphi$

By $\sqrt{}$ is understood the positive square root. In order that the given and the reduced integral may agree in sign, it is necessary that in extracting the root given in the last column of this table that sign shall be taken which renders $(ad - bc)$ positive, $\frac{ax+b}{cx+d}$ being the root.

CHICAGO, AUGUST, 1903.

THE NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS HAVE INTEGRALS IN COMMON

By A. B. Pierce

In the theory of the linear homogeneous differential equations, many properties and relations are found which are very closely parallel to those of the ordinary algebraic equations. One of these is the necessary and sufficient condition (as given in determinant form) that two such equations should have integrals in common.

The necessity and sufficiency of this condition have been established by von Escherich* and Heffter,† the latter of whom has also proved the conditions for multiple common integrals.‡ The method used in the present paper for establishing the sufficiency of this condition follows entirely new lines and has many points of principle in common with that for the corresponding algebraic problem as given by Mansion in his *Elements de la theorie des determinants.*§

In §1, I have proved the necessary condition after the method of von Escherich, which is the analogue of Sylvester's dyalitic method of elimination as used in the algebraic problem. The proof of the main problem in §2 is divided into two distinct parts: in the first part is the proof of a theorem of considerable generality and value aside from the purposes of this paper, which expresses a sufficient condition in symbolic form; in the second part, after a preliminary definition, and discussion and proofs of the properties of

* *Denkschriften der Wiener Academie*, vol. 46 (1883).

† *Crelle*, vol. 116 (1896) and *Archiv der Mathematik und Physik*, ser. 3, vol. 3 (1902).

‡ *Archiv. der Mathematik und Physik*, l. c.

§ Mansion: *Elements de la theorie des determinants*, sixth edition, Paris, 1900.

matrices involved, the sufficiency of the condition of §1 is proved on the basis of the above mentioned theorem. The form of proof here given was adopted to avoid the consideration of certain special cases which would otherwise require separate discussion, and also to make the transition to the cases of §3 simple and direct. In §3, I have proved necessary and sufficient conditions for the case of several independent integrals in common.

It is also to be noticed that the discussion here given, gives the results for the algebraic problem by using the differential equations with constant coefficients, as will be pointed out more in detail below.

We shall indicate by the symbol A the linear homogeneous differential operator

$$a_0 \frac{d^a}{dx^a} + a_1 \frac{d^{a-1}}{dx^{a-1}} + \cdots + a_{a-1} \frac{d}{dx} + a_a$$

of order a, in which the coefficients are functions of x; and also by Ay the result obtained by applying the operator A upon the quantity y, namely:

$$a_0 \frac{d^a y}{dx^a} + a_1 \frac{d^{a-1} y}{dx^{a-1}} + \cdots + a_{a-1} \frac{dy}{dx} + a_a y \ .$$

It should be noticed that we assume that the function a_0 is not identically zero. Similarly B, R, S are linear homogeneous differential operators of orders b, r, s respectively and whose coefficients, also functions of x, may be indicated by β, ρ, σ respectively; with subscripts to correspond. Also RAy shall mean that A is first applied to y and then R applied to the result; similarly for SBy.

Throughout this paper, we shall take $a \geqq b$.

1. Now since any function which satisfies $Ay = 0$ will also satisfy $\frac{d}{dx} A(y) = 0$, by differentiating Ay successively with respect to x and each time placing the result equal to zero, and similarly for By, we obtain a system of equations which is satisfied by every integral common to the two differential equations $Ay = 0$ and $By = 0$. If we differentiate the first $(b-1)$ times, the second $(a-1)$ times and rearrange, we obtain a system of $(a+b)$ equations which must be satisfied by every function which satisfies both of the given equations. These equations will be linear and homogeneous in y and its first

$(a + b - 1)$ derivatives; if the terms are arranged according to the derivatives of descending orders:

$$a + b - 1,\ a + b - 2,\ a + b - 3,\ \cdots,\ a + 1,\ a,\ a - 1,\ \cdots,\ b + 1,\ b,\ \cdots,\ 1,\ 0,$$

the coefficients will be the elements of the following determinant, which we shall call $\Delta(x)$:

$$\begin{vmatrix}
a_0, & a_1 + (b-1)a_0', & a_2 + (b-1)a_1' + \frac{(b-1)(b-2)}{2}a_0'', & \cdots & & & & & & & a_a^{b-1} \\
0, & a_0, & a_1 + (b-2)a_0', & \cdots & & & & & & & a_a^{b-2} \\
\cdot & \cdot & \cdot & \cdots & & & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdots & & & & & & & \cdot \\
0, & 0, & 0, & \cdots, & a_0, & a_1 + a_0', & a_2 + a_1', & \cdots & \cdot, & a_a + a_{a-1}', & a_a' \\
0, & 0, & 0, & \cdots, & 0, & a_0, & a_1, & \cdots & \cdot, & a_{a-1}, & a_a \\
0, & 0, & 0, & \cdots, & 0, & 0, & 0, & \cdots, 0, \beta_0, & \cdot, & \beta_{b-1}, & \beta_b \\
0, & 0, & 0, & \cdots, & 0, & 0, & 0, & \cdots, \beta_0, \beta_1 + \beta_0', & \cdot, & \beta_b + \beta_{b-2}', & \beta_b' \\
\cdot & \cdot & \cdot & \cdots & & & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdots & & & & & & & \cdot \\
0, & \beta_0, & \beta_1 + (a-2)\beta_0', & \cdots & & & & & & & \beta_b^{a-2} \\
\beta_0, & \beta_1 + (a-1)\beta_0', & \beta_2 + (a-1)\beta_1' + \frac{(a-1)(a-2)}{2}\beta_0'', & \cdots & & & & & & & \beta_b^{a-1}
\end{vmatrix}$$

Under these conditions, we know from the application of determinants to linear homogeneous equations that, unless y is identically zero, the determinant formed of the coefficients must vanish.

This gives the following theorem:

THEOREM: *A necessary condition that $Ay = 0$ and $By = 0$ shall have at least one integral, other than zero, in common is that $\Delta(x)$, as an expression in x, shall be identically zero.*

2. LEMMA: *If $A(y)$ and $B(y)$ be two linear homogeneous differential expressions such that we can find two linear homogeneous differential operators R and S for which $RAy \equiv SBy$ and $s < a$, then the two differential equations $Ay = 0$ and $By = 0$ have at least one integral, other than zero, in common.*

From the condition of identity in the hypothesis, $r + a = s + b$ and therefore $r < b$ if $s < a$.

Let $y_{a,1}, y_{a,2}, \cdots, y_{a,a}$ be a fundamental system of integrals of the differential equation $Ay = 0$.

Since $Ay_{a,i} = 0$, and therefore $RAy_{a,i} = 0$, for $i = 1, 2, \cdots, a$, our identity gives $SBy_{a,i} = 0$ for the same system of values of i. As a result, we have the following two possibilities: either (1) one or more of the functions $y_{a,i}$ is an integral of the equation $By = 0$; or (2) $y_{a,i}$, when substituted for y in the expression By, always gives an integral, not identically zero, of the differential equation $Sy = 0$.

The first of these possibilities establishes the theorem stated without further discussion, so we turn to the second alternative.

Let us indicate the results of the substitution as follows:

$$By_{a,1} = y_{s,1}, \quad By_{a,2} = y_{s,2}, \cdots, \quad By_{a,s} = y_{s,s},$$
$$By_{a,s+1} = y_{s,s+1}, \cdots, \quad By_{a,a} = y_{s,a}.$$

Of these functions $y_{s,k}$ not more than s can be linearly independent, so we will consider that our integrals $y_{a1}, y_{a2}, \cdots, y_{aa}$ have been so arranged that all of the functions $y_{s,k}$ which are independent are included in the first s of the a given above. Then we can write

$$y_{s,s+l} = \lambda_{1,l} y_{s,1} + \lambda_{2,l} y_{s,2} + \cdots + \lambda_{s,l} y_{s,s}$$

for all values of l from 1 to $(a - s)$ inclusive, where the quantities λ are constants and not all zero.

Let us now form a solution $Y_{a,s+l}$ of the equation $Ay = 0$ in the same way out of $y_{a,1}, y_{a,2}, \cdots, y_{a,s}$, using the same constants λ, as follows:

$$Y_{a,s+l} = \lambda_{1,l} y_{a,1} + \lambda_{2,l} y_{a,2} + \cdots + \lambda_{s,l} y_{a,s}.$$

Then $BY_{a,s+l} = y_{s,s+l}$; but we also have $By_{a,s+l} = y_{s,s+l}$. By subtraction, we obtain

$$B(Y_{a,s+l} - y_{a,s+l}) = 0.$$

That is, $(Y_{a,s+l} - y_{a,s+l})$ is an integral of the equation $By = 0$. But it is an integral of the equation $Ay = 0$ since it is a linear combination, with

constant coefficients, of integrals of this equation. Moreover $Y_{a,s+l} - y_{a,s+l}$ is not identically zero since the functions $y_{a,i}$ are linearly independent. Therefore the theorem is proved.

We have also the following corollary:

COROLLARY. *The differential equations* $Ay = 0$ *and* $By = 0$ *have at least* $(a - s)$ *independent integrals in common.*

$Y_{a,s+l}$ contains only the integrals $y_{a,1}, y_{a,2}, \cdots, y_{a,s}$, and the integrals $y_{a,s+l}$ are independent integrals for all the different values of l from 1 to $(a - s)$ inclusive. Therefore the expressions $Y_{a,s+l} - y_{a,s+l}$ are linearly independent of each other for the above set of values for l. But they are all integrals of the two equations. Therefore our statement is proved.

In proving the sufficiency of the condition $\Delta(x) \equiv 0$ by means of the lemma just proved, certain complications arise on account of the great number of ways in which this determinant may vanish: for example, by the vanishing of all its minors down to a certain definite order; or the vanishing of all the minors which enter matrices chosen according to certain laws, matrices to which I have given the special name, *submatrices* of the various orders, and which are defined as follows:

DEFINITION: The n^{th} submatrix of the determinant $\Delta(x)$ is obtained by cutting off the first n columns, the first n rows, and the last n rows.

We now prove three properties of these submatrices which are fundamental in our discussion.

PROPERTY I: If the n^{th} submatrix vanish, and $n < b$, then $\Delta(x)$ vanishes.

For the matrix obtained by cutting off the first n and also the last n rows of the determinant $\Delta(x)$ vanishes since the columns of this matrix in which all the elements are not zero are the columns of the n^{th} submatrix. Now expanding Δ according to its minors found in this matrix, we find that every term of this expansion has one factor zero and therefore $\Delta(x)$ must be zero.

PROPERTY II: If the $(n + 1)^{\text{st}}$ submatrix does not vanish, not all the first minors of the n^{th} submatrix can vanish.

Let D be a determinant of $(n + 1)^{\text{st}}$ submatrix which does not vanish. Construct the determinant of order one higher than D in which the first element of the first column is a_0 and all the other elements of that column are zeros and D is the minor of a_0, and also all the other elements of the first row are the elements of the first row of the n^{th} submatrix which occur in the columns with the elements of the columns of D. Then this new determinant is a first minor

of the n^{th} submatrix and being equal to $a_0 D$ is not identically zero: thus proving our property.

PROPERTY III: In $\Delta(x)$, there is always a submatrix which does not vanish if $Ay \not\equiv f(x) \cdot By$.

If $a > b$, the b^{th} submatrix is such a one; its first determinant being $\beta_0^{a-b} \not\equiv 0$. If $a = b$, the $(b-1)^{\text{st}}$ is such. Here the matrix has two rows and if it vanished we should have the relation

$$a_0 : \beta_0 = a_1 : \beta_1 = \cdots = a_k : \beta_k = \cdots,$$

which is excluded by the hypothesis.

We now take up our principal theorem.

THEOREM: *If $Ay = 0$ and $By = 0$ be two linear homogeneous differential equations of orders a and b whose coefficients make $\Delta(x)$ vanish identically, then $Ay = 0$ and $By = 0$ have at least one integral in common, which is not identically zero.*

We may assume that the first members of the differential equations are not proportional to each other as otherwise these equations would obviously have all their solutions in common. By Property III there exists therefore a submatrix of $\Delta(x)$ which does not vanish. Let the $(n+1)^{\text{st}}$ submatrix be the first one which does not vanish. It follows from the proof of Property III that $n < b$, or if $a = b$ that $n < b - 1$. Then, if $h = a + b - 2n$, it has $(h-2)$ rows and $(h+n-1)$ columns, and the n^{th} submatrix has h rows and $(h+n)$ columns. Writing the n^{th} submatrix, we have

$$\left\| \begin{array}{cccccccc}
a_0, & a_1 + (b-n-1)a_0', & \cdots & & & & , & a_a^{b-n-1} \\
0, & a_0, & \cdots & & & & , & a_a^{b-n-2} \\
\cdot & \cdot & \cdots & & & & & \cdot \\
0, & 0, & \cdots & 0, & a_0, & \cdots & , & a_a \\
0, & 0, & \cdots & & \beta_0, \beta_1, & \cdots & , & \beta_b \\
\cdot & \cdot & \cdots & & & & & \cdot \\
0, & \beta_0, & \cdots & & & & , & \beta_b^{a-n-2} \\
\beta_0, & \beta_1 + (a-n-1)\beta_0', & \cdots & & & & , & \beta_b^{a-n-1}
\end{array} \right\| \equiv 0$$

We shall first assume that the determinants of order $(h-1)$ given by the first $(h-1)$ columns are *not* all zero.

Now form all the determinants of order h in which the first $(h-1)$

columns are the first $(h-1)$ columns of the n^{th} submatrix and the h^{th} column is successively one of the remaining $(n+1)$ columns. In each of these determinants multiply the elements of the last column by one of the factors $\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \cdots, \frac{dy}{dx}, y$, according as they belong to the h^{th}, $(h+1)^{\text{st}}, \cdots,$ $(h+n-1)^{\text{st}}$, $(h+n)^{\text{th}}$, column of the submatrix respectively; and then add them all together. We thus obtain a new determinant in which every element of the last column is a linear homogeneous differential expression of order n. This new determinant vanishes because by hypothesis each of the determinants which we added together to form it vanished.

In the new determinant, we multiply the successive columns beginning with the first and including the $(h-1)^{\text{st}}$ by the factors $\frac{d^{h+n-1}y}{dx^{h+n-1}}, \frac{d^{h+n-2}y}{dx^{h+n-2}}, \cdots, \frac{d^{n+1}y}{dx^{n+1}}$ respectively and add to the last column. Placing $r=(b-n-1)$ and $s=(a-n-1)$, the relation can then be rewritten as follows:

$$\begin{vmatrix}
a_0, & a_1+ra_0', & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, & \frac{d^r}{dx^r}Ay \\
0, & a_0, & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, & \frac{d^{r-1}}{dx^{r-1}}Ay \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & \cdots, & 0, & a_0, & (a_1+a_0'), & \cdots & \cdots & \cdots & \cdots & \cdots, & \frac{d}{dx}Ay \\
0, & 0, & \cdots & 0, & a_0, & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, & Ay \\
0, & 0, & \cdots & \cdots & \cdots & \cdots & 0, & \beta_0, & \cdots & \cdots & \cdots, & By \\
0, & 0, & \cdots & \cdots & \cdots & \cdots & \cdots, & \beta_0, & (\beta_1+\beta_0), & \cdots & \cdots, & \frac{d}{dx}By \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0, & \beta_0, & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, & \frac{d^{s-1}}{dx^{s-1}}By \\
\beta_0, & \beta_1+s\beta_0', & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, & \frac{d^s}{dx^s}By
\end{vmatrix} \equiv 0.$$

If we expand this determinant according to the elements of the last column, we have the form

$$\rho_0 \frac{d^r}{dx^r} Ay + \rho_1 \frac{d^{r-1}}{dx^{r-1}} Ay + \cdots + \rho_{r-1} \frac{d}{dx} Ay + \rho_r Ay + \cdots$$
$$- \left\{ \sigma_s By + \sigma_{s-1} \frac{d}{dx} By + \cdots + \sigma_1 \frac{d^{s-1}}{dx^{s-1}} By + \sigma_0 \frac{d^s}{dx^s} By \right\} \equiv 0.$$

In this scheme, we notice, that, starting from the right, the elements of the first column of the submatrix are multiplied by y, those of the second column by $\frac{dy}{dx}$, those of the third by $\frac{d^2y}{dx^2}$, and so on.

It may happen that the particular $(h-1)$ columns, which we have picked out of the submatrix do not fulfil the hypothesis explicitly made concerning them and that in fact the quantities ρ_i and σ_j all vanish identically. But from Property II, we see that we can choose $(h-1)$ columns which do fulfil this hypothesis. Then we may proceed in a similar manner, always associating the elements of every column with the derivative of y of proper order.

Therefore we always have an expansion like that given above in which the quantities ρ_i and σ_j do not all vanish. Indeed from Property II, we see that the $(h-1)$ columns can be chosen so that neither ρ_0 nor σ_0 is zero.

Writing

$$\rho_0 \frac{d^r}{dx^r} + \rho_1 \frac{d^{r-1}}{dx^{r-1}} + \cdots + \rho_{r-1} \frac{d}{dx} + \rho_r \equiv R$$

and

$$\sigma_0 \frac{d^s}{dx^s} + \sigma_1 \frac{d^{s-1}}{dx^{s-1}} + \cdots + \sigma_{s-1} \frac{d}{dx} + \sigma_s \equiv S,$$

our relation becomes

$$RAy - SBy \equiv 0.$$

By the lemma of this section this gives us at least one integral common to $Ay = 0$ and $By = 0$, because $s = a - n - 1$ is less than a even for the most unfavorable case, $n = 0$.

Therefore under all possibilities, the identical vanishing of $\Delta(x)$ is a sufficient condition that $Ay = 0$ and $By = 0$ shall have an integral not identically zero in common.

We also have the following

COROLLARY: If the n^{th} submatrix vanishes, then $Ay = 0$ and $By = 0$ have at least $(n + 1)$ independent integrals in common.

3. After defining *principal minors* of $\Delta(x)$, we shall take up the discussion of necessary and sufficient conditions for multiple common integrals.

DEFINITION. The k^{th} *principal minor* of $\Delta(x)$ is that determinant obtained by cutting off the first k and the last k rows and columns, and will be represented by $\Delta_k(x)$.

THEOREM. *If $Ay = 0$ and $By = 0$ have $(m + 1)$ independent integrals in common, $\Delta(x)$ and its first m principal minors must vanish identically.*

Take the system of equations obtained by cutting off the first i and also the last i equations of the system from which $\Delta(x)$ is obtained in §1, when arranged as there indicated. This will leave $(a + b - 2i)$ equations in which the equation of highest order is of order $(a + b - i - 1)$. Eliminating the $(a + b - 2i - 1)$ derivatives of y of highest order, we shall finally obtain a linear homogeneous differential equation in y which is at most of order i. The coefficients of the various derivatives of y in this equation are determinants of the i^{th} submatrix of $\Delta(x)$, each of which contains the first $(a + b - 2i - 1)$ columns of the i^{th} submatrix as columns. The coefficient of $\frac{d^i y}{dx^i}$ in this equation is $\Delta_i(y)$.

This new equation of order i will be satisfied by the $(m + 1)$ independent common integrals of $Ay = 0$ and $By = 0$. Therefore if $i \leqq m$, every one of its coefficients must be zero; and so $\Delta_i(x)$ must vanish identically for $i = 1, 2, \cdots, m$.

THEOREM. *If $\Delta(x)$ and its first m principal minors vanish identically $Ay = 0$ and $By = 0$ have at least $(m + 1)$ independent integrals in common.*

We prove (1) that if $\Delta(x)$ and $\Delta_1(x)$ vanish identically, then $Ay = 0$ $By = 0$ have at least two independent integrals in common. Then in order to use the method of complete induction, we prove (2) that if there are k independent common integrals when $\Delta(x)$ and $\Delta_i(x)\,[i = 1, 2, \cdots, (k - 1)]$ vanish identically, there are at least $(k + 1)$ such integrals when $\Delta_k(x)$ also vanishes identically.

First. Let $\Delta(x)$ and $\Delta_1(x)$ vanish identically: further let the $(n_1 + 1)^{\text{st}}$ submatrix of $\Delta_1(x)$ be the first one which does not vanish. If now we multiply the elements of the various columns of the n_1^{th} submatrix by derivatives of y selected as indicated in the proof of the theorem of §2, and add together

in a properly chosen column as before, we shall obtain a determinant which must vanish identically. Since the elements of the last column of $\Delta(x)$ do not appear in $\Delta_1(x)$, we shall find that y does not appear as a term of this new determinant, although its derivatives do. Writing $b - n_1 - 2 = r_1$ and $a - n_1 - 2 = s_1$, the column containing the derivative of y can be written

$$\frac{d^{r_1}}{dx^{r_1}}Ay - A_{0,b-2}\,y, \frac{d^{r_1-1}}{dx^{r_1-1}}Ay - A_{0,b-3}\,y, \cdots, \frac{d}{dx}Ay - A_{0,1}\,y, Ay - A_{0,0}\,y,$$

$$By - B_{00}\,y, \frac{d}{dx}By - B_{01}y, \cdots, \frac{d^{s_1-1}}{dx^{s_1-1}}By - B_{0,a-3}\,y, \frac{d^{s_1}}{dx^{s_1}}By - B_{0,a-2}\,y,$$

where $A_{0,i}$, B_{0j} represent functions of x which are coefficients of y, or linear homogeneous differential operators of order zero. Expanding the determinant according to this column, we obtain a relation of the form

$$R_1Ay - S_1By - T_0y \equiv 0,$$

where R_1, S_1, T_0 are linear homogeneous differential operators of orders r_1, s_1, and 0 respectively. Substituting a common integral Y, not identically zero, of $Ay = 0$ and $By = 0$ in this relation, we have $T_0Y = 0$. Therefore T_0 vanishes identically and the above relation becomes

$$R_1Ay - S_1By \equiv 0.$$

Then, from the lemma, $Ay = 0$ and $By = 0$ must have at least $(n_1 + 2)$ and therefore at least two independent integrals in common.

Secondly. Let $\Delta(x)$ and $\Delta_i(x)$ $(i = 1, 2, \cdots, k)$ vanish identically: also let the $(n_k + 1)^{st}$ submatrix of $\Delta_k(x)$ be the first one which does not vanish.

As before, we multiply the elements of the various columns of the n_k^{th} submatrix of $\Delta_k(x)$ by the derivatives of y of proper order and add together in a properly selected column. There results a determinant which must vanish identically.

Since the elements of the last k columns of Δx do not enter $\Delta_k(x)$, y and its derivatives of order lower than k do not appear in this new determinant. Writing $(b - n_k - k - 1) = r_k$ and $(a - n_k - k - 1) = s_k$ the elements of this column can be written

$$\frac{d^{r_k}}{dx^{r_k}} Ay - A_{k-1,\,b-k-1}\, y,\ \frac{d^{r_k-1}}{dx^{r_k-1}} Ay - A_{k-1,\,b-k-2}\, y,\ \cdots,$$

$$\frac{d}{dx}Ay - A_{k-1,1}\, y,\ Ay - A_{k-1,0}y,\ By - B_{k-1,0}\, y, \frac{d}{dx}By - B_{k-1,1}\, y, \cdots,$$

$$\frac{d^{s_k-1}}{dx^{s_k-1}} By - B_{k-1,\,a-k-2}\, y, \frac{d^{s_k}}{dx^{s_k}} By - B_{k-1,\,a-k-1}\, y,$$

where $A_{k-1,i}$, $B_{k-1,\,i}$ represent linear homogeneous differential operators of order $(k-1)$. Expanding according to this column we obtain a relation of the form

$$R_k Ay - S_k By - T_{k-1}\, y \equiv 0,$$

where R_k, S_k, T_{k-1} are linear homogeneous differential operators of order r_k, s_k, and $(k-1)$ respectively. Substituting k independent integrals Y_1, $Y_2 \cdots, Y_k$ common to $Ay = 0$ and $By = 0$, which by hypothesis they must have on account of the identical vanishing of $\Delta(x)$ and $\Delta_i(x)$ $[i = 1, 2, \cdots, (k-1)]$, we find that the differential equation of $(k-1)^{st}$ order,

$$T_{k-1}\, y = 0,$$

has k independent integrals. Therefore $T_{k-1}\, y$ vanishes identically and our relation becomes

$$R_k Ay - S_k By \equiv 0.$$

From the lemma, $Ay = 0$ and $By = 0$ must have at least $(n_k + k + 1)$ and therefore at least $(k+1)$ independent integrals in common.

Our proof is now completed by the method of mathematical induction.

Combining the last two results we have the

THEOREM: *A necessary and sufficient condition that two linear homogeneous differential equations $Ay = 0$ and $By = 0$ shall have k independent integrals in common is that $\Delta(x)$ and its first $(k-1)$ principal minors shall vanish identically.*

This gives the

COROLLARY: *Two linear homogeneous differential equations $Ay = 0$ and $By = 0$ have k and only k independent integrals in common, if and only if $\Delta(x)$ and its first $(k-1)$ principal minors vanish identically and the k^{th} principal minor does not.*

4. I stated in the introduction that this discussion would give us all the results for the corresponding problem of algebra. I shall now justify that statement.

First, if we take the differential equation with constant coefficients and follow the process given for the necessity of the condition, we shall find that the resulting formula is just what we should obtain if we put λ^k for $\frac{d^k y}{dx^k}$ and $\lambda^0 = 1$ for y and then carried out the investigation as is ordinarily done in algebra.

Further we know that the solutions of linear homogeneous differential equations with constant coefficients are in general of the form $e^{\lambda x}$, where λ must satisfy the equation obtained above. From this consideration, we see clearly why we should obtain the solution of the algebraic problem as a special case from that of the differential equation.

5. The following particular cases are given by way of illustration.

1. Take

$$Ay \equiv \frac{d^3y}{dx^3} + (x - a)\frac{d^2y}{dx^2} + x(x - a)\frac{dy}{dx} - ax^2y,$$

$$By \equiv \frac{d^2y}{dx^2} + (x - a)\frac{dy}{dx} - axy.$$

It can easily be verified that if R and S be taken as follows:

$$R \equiv (1 - x^2)\frac{d}{dx} + x(3 - x^2),$$

$$S \equiv (1 - x^2)\frac{d^2}{dx^2} + x(3 - x^2)\frac{d}{dx} - (1 - x^2)^2,$$

then

$$RAy \equiv SBy.$$

Therefore $Ay = 0$ and $By = 0$ have at least one common integral not identically zero, since $a - s = 1$. By substituting e^{ax} for y in the two differential equations we find them to be satisfied.

2. Take

$$Ay \equiv \frac{d^3y}{dx^3} + x(x + 1)\frac{d^2y}{dx^2} + (x^3 + x^2 + 1)\frac{dy}{dx} + x(x^3 + 2)y,$$

$$By \equiv \frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + (2x^2 + 1)\frac{dy}{dx} + x(x^2 + 2)y,$$

and

$$R \equiv x(x-1)\frac{d}{dx} + (x^3 - x^2 - 2x + 1),$$

$$S \equiv x\,(x-1)\frac{d}{dx} + (x^4 - x^3 - 2x + 1).$$

Then

$$RAy \equiv SBy.$$

From the lemma, we conclude that the two equations $Ay = 0$ and $By = 0$ have at least two independent integrals in common, since $a - s = 2$. In fact, the common integrals are the solutions of the differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0.$$

3. Take

$$Ay \equiv \frac{d^4y}{dx^4} + x\frac{d^3y}{dx^3} + (x^2-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - x^2y,$$

$$By \equiv \frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - \frac{dy}{dx} - x^2y,$$

and

$$R \equiv x(x^3 - 2x^2 + x - 1)\frac{d}{dx} + (x^6 - 2x^5 + 4x^4 - 7x^3 - 2x + 2),$$

$$S \equiv x\,(x^3 - 2x^2 + x - 1)\frac{d^2}{dx^2} + (x^5 + x^4 - 5x^3 - x^2 - 2x + 2)\frac{d}{dx}$$

$$+ (-2x^6 + 2x^5 + 2x^4 - x^3 - x^2 + 3x).$$

We can very easily verify that $Ay = 0$ and $By = 0$ have e^x and e^{-x} as common solutions, *i. e.* the solutions of the differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

In these illustrations, we find that the number of independent integrals which the two equations have in common is exactly $(a - s)$. The reason for this is that the expressions given for R and S are those of the lowest order for which the identity is true.

ANN ARBOR, MICHIGAN,
DECEMBER 1903.

A GENERAL METHOD OF EVALUATING DETERMINANTS

By G. Macloskie

The method formerly published (Annals of Mathematics, January 1900, p. 74) may be further simplified if we start from the following theorems:

$$\begin{vmatrix} a & b & c & \cdots \\ e & f & g & \cdots \\ j & k & l & \cdots \\ \cdots & & & \end{vmatrix} = \frac{1}{a^{n-1}} \begin{vmatrix} a & ab & ac & \cdots \\ e & af & ag & \cdots \\ j & ak & al & \cdots \\ \cdots & & & \end{vmatrix} = \frac{1}{a^{n-2}} \begin{vmatrix} af-be & ag-ce & \cdots \\ ak-bj & ab-cj & \cdots \\ \cdots & & \end{vmatrix}$$

We thus obtain the general rule:

1. Line off a superior border row and an anterior border column of the given determinant of order n.

2. Multiply the leading element a into the several elements of its minor, deducting from each product the product of the two border elements on the same row and column.

The results are the elements of a new determinant, which on being divided by a^{n-2} is equal to the original determinant.

We may often find it convenient to shift the lines so as to bring a unit element into the leading place. When zero appears in the leading place, shifting is necessary.

Example:

$$\left|\begin{array}{cccc} 2 & 7 & -2 & 8 \\ \hline 4 & 1 & 1 & -3 \\ 0 & 3 & -1 & 4 \\ 6 & 4 & 2 & -8 \end{array}\right| = \frac{1}{2^2} \begin{vmatrix} -26 & 10 & -38 \\ 6 & -2 & 8 \\ -34 & 16 & -64 \end{vmatrix}$$

$$= 2 \left|\begin{array}{ccc} 13 & -5 & 19 \\ \hline 3 & -1 & 4 \\ 17 & -8 & 32 \end{array}\right| = \frac{2}{13} \begin{vmatrix} 2 & -5 \\ -19 & 93 \end{vmatrix} = 14.$$

Princeton University,
February, 1904.

APPLICATION OF GROUPS TO A COMPLEX PROBLEM IN ARRANGEMENTS

By L. E. Dickson

1. Viewed as a problem in configurations, the general problem here considered possesses especial interest. The application of group-theory to problems of this type affords not only a better insight into the essential features of the problem but also a method for the rapid construction of solutions and, what is even more essential in the longer solutions, affords simultaneously a proof of their correctness.

In its initial form* the problem† related to the seating of m persons at a round table on successive days such that no one shall sit twice between the same two companions. The precise formulation of the general problem will be preceded by some definitions and explanations and the complete consideration of the example $m = 5$.

While we are concerned with a ring and not a row, nevertheless we shall, for convenience in printing, write the symbols in a straight row and not in ring form. Then in $ABCDE$ it is understood that E is followed by A. It being a question of relative order only, we regard as *equivalent* a ring and the one derived from it by reversing the order of its symbols. Then, for $m = 5$, the following ten arrangements are equivalent:

$$ABCDE,\quad BCDEA,\quad CDEAB,\quad DEABC,\quad EABCD$$
$$AEDCB,\quad BAEDC,\quad CBAED,\quad DCBAE,\quad EDCBA.$$

In general, the $2m$ arrangements derived from a given one by applying the substitutions of the dihedron group G_{2m} are regarded as *equivalent*. The $2m$

* Proposed in the *American Mathematical Monthly*, 1899, p. 92, by Professor C. H. Judson, who gave examples for $m = 6$ and $m = 8$ in the *Monthly*, 1900, p. 72. Another example for $m = 6$ and one for $m = 7$ by Dr. F. H. Safford will appear soon in the *Monthly*. The two examples for $m = 6$, whose correctness I have verified, are here reproduced.

† I give an equivalent problem, to emphasize the notion of dihedrons: In each of a set of circular discs are bored m holes of different shape at the vertices of a concentric regular polygon (the latter the same for each disc). It is required to determine all sequences of the holes on the various discs such that it shall be impossible to place two discs in a position where three consecutive holes of the same shape are juxtaposed.

arrangements will be said to define the same *dihedron*,* which may be represented by any one of the $2m$ arrangements. However, we shall employ always one of the two arrangements which begins with a particular letter, as A.

Two dihedrons, as $ABCDE$ and $ABECD$, will be called *consistent* when there is no letter which occurs in both between the same two letters.

For $m=5$, the only dihedrons consistent with $ABCDE$ are

$$ABECD, \quad ABDEC, \quad ACBED, \quad ACDBE, \quad ADBCE \quad ACEBD.$$

Any two of the first five are consistent, while no one of them is consistent with the sixth.

A set of k mutually consistent dihedrons on m letters will be called a *complete set*, S_k^m, when every further dihedron on the m letters is inconsistent with at least one dihedron of S_k^m. Hence for $m=5$ the only complete sets are

$$S_6^5 = \{ABCDE,\ ABDEC,\ ABECD,\ ACBED,\ ACDBE,\ ADBCE\},$$

$$S_2^5 = \{ABCDE,\quad ACEBD\}.$$

The general problem is to find all complete sets S_k^m of dihedrons. The maximum value of k is $\frac{1}{2}(m-1)(m-2)$, this being the number of pairs of letters between which a given one may lie. For this maximum, k, a set S_k^m possesses remarkable symmetry. It may be given a form, using the symbols $A_1, \cdots, A_m$, such that there are exactly $m-i$ dihedrons $A_1 A_2 \cdots$, for $i=2, \cdots, m-1$. This is exemplified in S_6^5 above and in the following complete sets S_{10}^6:

(Judson)	(Safford)	(Dickson)
$ABCDEF$	$ABCDEF$	$ABCDEF$
$ABDCFE$	$ABDCFE$	$ABDFEC$
$ABEFDC$	$ABEDFC$	$ABECFD$
$ABFECD$	$ABFECD$	$ABFDCE$
$ACFDBE$	$ACDFBE$	$ACDBFE$
$ACEBFD$	$ACEBDF$	$ACFEBD$
$ACBEDF$	$ACBEFD$	$ACBEDF$
$ADECBF$	$ADECBF$	$ADECBF$
$ADBFCE$	$ADBFCE$	$ADCFBE$
$AEDBCF$	$AEDBCF$	$AEDBCF$

* This corresponds to the dihedron in geometry, that is, a regular polygon whose plane has two aspects, with order attached to the vertices. If, on viewing the polygon from a point above its plane, the vertices are read in a certain order, then from a point below the plane they will appear in the reverse order.

In view of the particular interest of the sets with maximum k, we will be concerned here* with *the special problem to find all maximum complete sets* S_k^m, *where* $k = \frac{1}{2}(m-1)\ (m-2)$.

2. Before undertaking in general the question of the applicability of group theory to the present problems, we will consider two of the simplest cases.

For $m=4$ the distinct dihedrons are $ABCD$, $ABDC$, $ACBD$, and these are consistent. They are equivalent to $ADCB$, $ACDB$, $ADBC$, respectively, each having the symbols of the corresponding one in reverse order. The six give all permutations of B, C, D, and hence define the symmetric group on three letters.

In S_6^5 the dihedrons are obtained from $ABCDE$ by the substitutions:

$$I = \text{identity},\ (CDE),\ (CED),\ (BC)(ED),\ (BCD),\ (BDC).$$

These do not form a group since $(CDE)(BDC) = (BDE)$. Taking, however, the second notation $AEDCB, \cdots$, for each of the 6 dihedrons, we find that they are obtained from $ABCDE$ by the substitutions

$$(BE)(CD),\ (BCE),\ (BDE),\ (BD)(EC),\ (BEC),\ (BED).$$

These together with the preceding six substitutions form the alternating group on 4 letters. Extending it by $(ABCDE)$, to pass from rings to rows, we obtain† the alternating groups on 5 letters.

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* I have made considerable headway on the *general* problem for $m = 6$. The method begins as for $m = 5$, there being for $m = 6$ exactly 38 dihedrons consistent with $ABCDEF$. The number of cases is greatly reduced by transformation by suitable substitutions. The aim is to obtain a final list of S_k^6 in which no two differ by a mere change in notation, *i. e.* are not conjugate under the symmetric group on 6 letters. I give a few non-maximum complete sets:

$$S_8^6 = \{ABCDEF,\ ABDFEC,\ ABEFCD,\ ABFDCE,\ ACDBFE,\ ACFBED,\ ADBECF,\ ADFCBE\};$$

$$S_7^6 = \{ABCDEF,\ ABDFEC,\ ABECFD,\ ABFDCE,\ ACDBFE,\ ACFBED,\ AEDBCF\};$$

$$S_7^6 = \{ABCDEF,\ ABDCFE,\ ABECFD,\ ABFDEC,\ AECBDF,\ AFCBED,\ AEDBFC\};$$

$$S_7^6 = \{ABCDEF.\ ABDFEC,\ ABEFCD,\ ABFDCE,\ ACBEDF,\ ADBECF,\ ACDBFE\},$$

where for each S_7^6 there is a single complete set having the first six dihedrons.

† This might have been anticipated since of the two substitutions

$$\begin{pmatrix} ABCDE \\ \alpha\ \beta\ \gamma\ \delta\ \epsilon \end{pmatrix} = S, \begin{pmatrix} ABCDE \\ \alpha\ \beta\ \gamma\ \epsilon\ \delta \end{pmatrix} = S(\delta\epsilon),$$

one is even and one odd.

3. For the general case of $S^m_{\frac{1}{2}(m-1)(m-2)}$, we likewise take each dihedron in direct and reverse order, so that we have $(m-1)(m-2)$ arrangements (rings) beginning with a fixed letter A. The group $G_{(m-1)(m-2)}$ of the resulting substitutions must be doubly-transitive since we have the arrangements $AX\cdots Y$, where X and Y are arbitrary. Hence* $m-1$ is the power of a prime p^n. Doubly transitive groups of degree p^n and order $p^n(p^n-1)$ have been completely enumerated† for general p and $n=1, 2, 3$, and‡ for $p^n=2^4$, the result presumably§ holding also for $n>3$. There is a single type of doubly-transitive group $G^{p^n}_{p^n(p^n-1)}$, except for $p^n=3^2$ when there is a second type H^9_{72}. The first may be taken concretely as the group of all linear transformations

$$(1) \qquad S_{a,b}: \qquad z'=az+b \qquad (a\neq 0),$$

with coefficients in the Galois Field of order p^n. Allowing z to run through the series of marks of this $GF[p^n]$, we obtain a group on p^n letters, which is doubly-transitive since we may solve

$$z_1'=az_1+b, \qquad z_2'=az_2+b \qquad (z_1\neq z_2,\ z_1'\neq z_2')$$

for a and b in the field, $a\neq 0$. The additional type is¶

$$(2) \qquad H^9_{72}=\{(abcd)(efgh), \qquad (ahcf)(bgde), \qquad (iac)(bgh)(dfe)\}$$

Considering the m arrangements in each ring, we extend the preceding groups by the cyclic substitution $(A\cdots)$, and obtain triply-transitive groups. From G, we obtain the group of the $(p^n+1)\,p^n\,(p^n-1)$ linear fractional transformations

$$(3) \qquad z'=\frac{az+b}{cz+d} \qquad (ad-bc\neq 0).$$

From H^9_{72} we obtain Cole's group 720_3 of order 720 on 10 letters.** The

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* Jordan, Récherches sur les substitutions, *Journal de Mathématiques*, ser. 2, vol. 17, (1872), pp. 351–367. (*Cf.* next reference).

† Burnside, On doubly-transitive groups of degree n and order $n(n-1)$, *Messenger of Mathematics*, vol. 25 (1895-96), pp. 147–153.

‡ Miller, *American Journal*, vol. 20 (1898), p. 231.

§ If further exceptions occur, they arise for $p^n\geqq 81$, so that their use in the present problem would be impracticable.

¶ Cole's (Ii), *Bulletin*, 1893, p. 257. His (Ii) is G^9_{72}.

** *Quarterly Journal*, vol. 27 (1895), p. 44. His 720_2 is the group of (3).

preceding types of doubly and triply transitive groups are all of the specified orders that occur in the lists of substitution groups on m letters, $m \geqq 17$.

4. *Application of the linear group.* Since the linear fractional group is triply-transitive, we may assume that the desired complete set, S_k^m, where $m = p^n + 1$ and $k = \frac{1}{2}p^n(p^n - 1)$, contains the dihedron*

$$(4) \qquad \infty \quad 0 \quad 1 \quad a_3 \quad a_4 \cdots a_{p^n},$$

where $a_1 = 0$, $a_2 = 1$, $a_3, a_4, \cdots, a_{p^n}$ give the p^n distinct marks of the $GF[p^n]$. Now $S_{r,s}$, defined by (1), replaces dihedron (4) by

$$(5) \qquad \infty \quad s \quad r+s \quad a_3 r + s \quad a_4 r + s \cdots a_{p^n} r + s.$$

We impose first the condition that there shall exist a linear transformation T, of the form (1), which replaces (4) by

$$(6) \qquad \infty \quad a_{p^n} \cdots a_4 \quad a_3 \quad 1 \quad 0,$$

the latter being a second notation for the dihedron (4). Hence, if $p > 2$,

$$(7) \quad T = (0\, a_{p^n})(1\, a_{p^n-1})(a_3 a_{p^n-2}) \cdots (a_i a_{p^n-i+1}) \cdots (a_{\frac{1}{2}(p^n-1)} a_{\frac{1}{2}(p^n+3)})$$

leaving fixed $a_{\frac{1}{2}(p^n+1)}$. If $p = 2$, the final transposition in T is $(a_{2^{n-1}}\, a_{2^{n-1}+1})$, and no letter remains fixed. Thus T is of period 2 and hence of the form

$$(8) \qquad T: \qquad z' = -z + a_{p^n}.$$

The conditions that (8) shall represent (7) are, since $a_2 = 1$,

$$(9) \qquad a_{p^n-i} = a_{p^n} - a_{i+1}, \qquad [i = 1, \cdots, q],$$

where $q = \frac{1}{2}(p^n - 1)$ if $p > 2$, $q = 2^{n-1} - 1$ if $p = 2$.

We impose next the condition that the dihedrons (5), with r, s, ranging over the $GF[p^n]$, $r \neq 0$, shall form a complete set. Taking two of these dihedrons, arising from $S_{r,s}$ and $S_{\rho,\sigma}$, we look for both forward and backward coincidences in triples. But if a linear transformation S replaces $\infty\, a_1 \cdots a_{p^n}$ by $\infty\, \beta_1 \cdots \beta_{p^n}$, and if $\{a_2, a_3, a_4\} \equiv \{\beta_3, \beta_2, \beta_1\}$, for example, then the product TS replaces $\infty\, a_1 \cdots a_{p^n}$ by $\infty\, \beta_{p^n} \cdots \beta_1$ so that for the latter $\{a_2, a_3, a_4\} \equiv \{\beta_1, \beta_2, \beta_3\}$. Hence, unless ∞ enters, we need only look for

* The element ∞ here adjoined to the field is used in the sense of *Linear Groups*, p. 260.

forward coincidences in triples. When ∞ enters, the only possible backward coincidence is

$$\{a_{p^n}\, r+s,\ \infty,\ s\} \equiv \{\sigma,\ \infty,\ a_{p^n}\,\rho+\sigma\},$$

requiring $\rho=-r$, $\sigma=a_{p^n}\,r+s$, whence

$$S_{\rho,\sigma}=TS_{r,s}.$$

But we retain only one of the arrangements resulting from the two transformations $S_{r,s}$ and $TS_{r,s}$. Hence there here results no condition on the a_i.

We proceed to examine the forward coincidences in triples. If

$$\{\rho,\ \rho+\sigma,\ a_3\rho+\sigma\} \equiv \{r+s,\ a_3 r+s,\ a_4 r+s\},$$

then $\rho=(a_3-1)r$, $a_3\rho=(a_4-1)r$. Then if

$$a_3(a_3-1)\neq a_4-1, \tag{10}$$

we would have $r=\rho=0$, $s=\sigma$, and the two dihedrons in question would be identical. Hence (10) is a sufficient condition that the specified coincidence in triples does not arise for distinct dihedrons. It is also a necessary condition. For if (10) does not hold, we may determine ρ and σ in terms of r and s, so that the coincidence shall arise. Likewise, to avoid a coincidence

$$\{\sigma,\ \rho+\sigma,\ a_3\rho+\sigma\} \equiv \{a_i r+s,\ a_{i+1}r+s,\ a_{i+2}r+s\}, \qquad (3\leqq i\leqq p^n-2)$$

we have the necessary and sufficient conditions

$$a_3(a_{i+1}-a_i)\neq a_{i+2}-a_i \qquad (i=3,\cdots,p^n-2). \tag{11}$$

To avoid the coincidence of $\{\rho+\sigma,\ a_3\rho+\sigma,\ a_4\rho+\sigma\}$ with

$$\{s,\ r+s,\ a_3 r+s\},\quad \{a_i r+s,\ a_{i+1}r+s,\ a_{i+2}r+s\},$$

we obtain, respectively, conditions (10) and

$$\begin{vmatrix} a_3-1 & a_{i+1}-a_i \\ a_4-1 & a_{i+2}-a_i \end{vmatrix} \neq 0 \qquad (i=3,\cdots,p^n-2). \tag{12}$$

To avoid the coincidence of $\{a_i\rho+\sigma,\ a_{i+1}\rho+\sigma,\ a_{i+2}\rho+\sigma\}$ with

$$\{s,\ r+s,\ a_3 r+s\},\ \{r+s,\ a_3 r+s,\ a_4 r+s\},\ \{a_j r+s,\ a_{j+1}r+s,\ a_{j+2}r+s\},$$

we obtain, respectively, conditions (11), (12), and

$$\begin{vmatrix} a_{i+1}-a_i & a_{j+1}-a_j \\ a_{i+2}-a_i & a_{j+2}-a_j \end{vmatrix} \neq 0 \qquad (i,j=3,\cdots,p^n-2\,;\ i<j). \tag{13}$$

Finally, since $0, 1, a_3, \cdots, a_{p^n}$ are the distinct marks of the field,

$$\{\infty, \sigma, \rho+\sigma\} \neq \{\infty, s, r+s\}, \quad \{a_{p^n}\rho+\sigma, \infty, \sigma\} \neq \{a_{p^n}r+s, \infty, s\},$$
$$\{a_{p^n-1}\rho+\sigma, a_{p^n}\rho+\sigma, \infty\} \neq \{a_{p^n-1}r+s, a_{p^n}r+s, \infty\}.$$

We have now established the following

THEOREM. *The distinct* dihedrons* (5) *derived from the dihedron* (4) *upon applying the transformations* (1) *of the linear group in the* $GF[p^n]$ *form a complete set* S_1^m, *where* $m = p^n+1$ *and* $k = \frac{1}{2}p^n(p^n-1)$, *if and only if conditions* (9), (10), (11), (12), *and* (13) *hold.*†

5. *The example* $p^n = 5$. The a_i are now integers taken modulo 5. Conditions (9) now give $a_4 \equiv a_5 - 1$, $a_3 \equiv 3a_5 \pmod 5$. Then (10) becomes

$$3a_5(3a_5-1) \neq 2(3a_5-1), \text{ i.e., } a_5 \neq 2 \text{ or } 4 \pmod 5.$$

But a_3, a_4, a_5 form a permutation of 2, 3, 4. Hence $a_5 \equiv 3$. Thus $a_3 \equiv 4$; $a_4 \equiv 2$. There is now a single condition (11) and a single (12), each given by $i = 3$; but no (13). The first two reduce to $-3 \neq -1$, $-1 \neq 0$. Hence for $p^n = 5$, group theory furnishes one and but one complete set‡ S_{10}^6:

$S_{1,0}$:	∞	0	1	4	2	3		$S_{1,1}$:	∞	1	2	0	3	4
$S_{4,0}$:	∞	0	4	1	3	2		$S_{4,1}$:	∞	1	0	2	4	3
$S_{2,0}$:	∞	0	2	3	4	1		$S_{3,4}$:	∞	4	2	1	0	3
$S_{3,0}$:	∞	0	3	2	1	4		$S_{1,4}$:	∞	4	0	3	1	2
$S_{2,1}$:	∞	1	3	4	0	2		$S_{2,4}$:	∞	2	4	0	1	3

It is immediately identified with Judson's solution (§ 1).

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* Employing only one of each pair resulting from $S_{r,s}$, $TS_{r,s}$.

† We do not discuss in general the question of the existence of at least one set of distinct marks a_i satisfying the specified conditions. As shown in the examples, such sets exist for $p^n = 5, 7, 9, 11$, and their number increases rapidly with p^n. By §§ 1, 2, there is a single set for $p^n = 3$ or 4, as may now be verified directly. For example, for $p^n = 2^2$, the marks are $0, 1, j, j+1$, where $j^2 \equiv j+1 \pmod 2$. Since $j+1$ satisfies the same congruence, we may by choice of the notation, set $a_3 = j$, $a_4 = j+1$. Then (10) and the single condition (9) are satisfied, while (11), (12), (13) do not occur.

‡ It is transformed into a complete set with the first dihedron ∞ 0 1 2 3 4 by (2 3 4). The analytic representation of (2 3 4) is

$$z' \equiv 3z^3 - z^2 - z \pmod 5.$$

The group of the resulting complete set is therefore not linear.

6. *The example* $p^n = 7$. Now a_3, a_4, a_5, a_6, a_7 form a permutation of 2, 3, 4, 5, 6. Conditions (9) give

$$(9') \qquad a_6 \equiv a_7 - 1, \quad a_5 \equiv a_7 - a_3, \quad a_4 \equiv 4a_7 \pmod 7.$$

Conditions (11) for $i = 3, 4, 5$ become (modulo 7)

$$\begin{aligned} a_3(4a_7 - a_3) &\neq a_7 - 2a_3, \quad \text{or} \quad (4a_3 - 1)(a_7 - 2a_3) \neq 0, \\ a_3(4a_7 - a_3) &\neq 4a_7 - 1, \quad \text{or} \quad (a_3 - 1)(4a_7 - a_3 - 1) \neq 0, \\ a_3(a_3 - 1) &\neq a_3, \quad \text{or} \quad a_3(a_3 - 2) \neq 0. \end{aligned}$$

Hence $a_3 \neq 2$, $a_7 \neq 2a_3$, $a_7 \neq 2a_3 + 2$. The three conditions (12) become

$$(4a_7 - a_3)(a_3 - 2a_7 + 3) \neq 0, \quad (4a_7 - 1)(a_3 - 2a_7 + 3) \neq 0,$$
$$(a_3 - 1)(a_3 - 4a_7 + 1) \neq 0,$$

whence $a_7 \neq 4a_3 + 5$, $a_7 \neq 2$. The three conditions (13) become

$$(4a_7 - a_3)(3a_7 + 2a_3 - 1) \neq 0, \quad (4a_7 - a_3)(3a_3 + 1) \neq 0,$$
$$4a_7 - a_3^2 + a_3 - 1 \neq 0,$$

the last being (10). Hence the only conditions on the a_i are (9'), (10), and

$$a_3 \neq 2, \quad a_7 \neq 2, \quad a_7 \neq 2a_3 \quad a_7 \neq 2a_3 + 2, \quad a_7 \neq 4a_3 + 5.$$

If $a_3 = 3$, then a_7 differs from 0, 1, 2, 3, 6, while $a_7 = 4$ gives $a_6 = 3 = a_3$ and is excluded. Proceeding similarly for $a_3 = 4, 5, 6$, we obtain only the following admissible systems a_i:

a_3	a_4	a_5	a_6	a_7
3	6	2	4	5
4	3	2	5	6
6	5	4	2	3
6	2	5	3	4

The second system leads to the complete set* S_{21}^8 :

* It is immediately identified with Judson's example for 8 persons.

	∞	0	1	4	3	2	5	6
$S_{4,0}$:	∞	0	4	2	5	1	6	3
$S_{3,0}$:	∞	0	3	5	2	6	1	4
$S_{2,0}$:	∞	0	2	1	6	4	3	5
$S_{5,0}$:	∞	0	5	6	1	3	4	2
$S_{6,0}$:	∞	0	6	3	4	5	2	1
$S_{4,1}$:	∞	1	5	3	6	2	0	4
$S_{3,1}$:	∞	1	4	6	3	0	2	5
$S_{2,1}$:	∞	1	3	2	0	5	4	6
$S_{5,1}$:	∞	1	6	0	2	4	5	3
$S_{6,1}$:	∞	1	0	4	5	6	3	2

$S_{2,4}$:	∞	4	6	5	3	1	0	2
$S_{5,4}$:	∞	4	2	3	5	0	1	6
$S_{1,4}$:	∞	4	5	1	0	6	2	3
$S_{6,4}$:	∞	4	3	0	1	2	6	5
$S_{5,3}$:	∞	3	1	2	4	6	0	5
$S_{1,3}$:	∞	3	4	0	6	5	1	2
$S_{4,3}$:	∞	3	0	5	1	4	2	6
$S_{4,2}$:	∞	2	6	4	0	3	1	5
$S_{3,2}$:	∞	2	5	0	4	1	3	6
$S_{6,5}$:	∞	5	4	1	2	3	0	6

The remaining three systems furnish complete sets different from the preceding. Whether they are themselves distinct is not discussed here.

7. *The example* $p^n = 3^2$. As the marks a_i of the $GF[3^2]$, we may take $0, \pm 1, \pm i, \pm i + 1, \pm i - 1$, where $i^2 \equiv -1 \pmod 3$. Then the only systems of solutions a_i of (9), (10), (11), (12), (13) are found to be the four:

a_3	a_4	a_5	a_6	a_7	a_8	a_9
$\pm i + 1$	$\mp i - 1$	$\pm i - 1$	-1	$\pm i$	$\mp i$	$\mp i + 1$
$\pm i + 1$	$\mp i - 1$	$\mp i$	$\mp i + 1$	-1	$\pm i - 1$	$\pm i$

8. *Application of the isolated group* H_{72}^9. This group (2) has a self-conjugate subgroup G of order 9 composed of the substitutions C, D, E, F (given below) and their powers. We may form a table of the distinct substitutions H_{72}^9 with those of G in the first row (here printed as first column), the left-hand multipliers being I = identity, A, $A^2 = B^2$, A^3, B, B^3, AB, $BA = A^3B = (AB)^{-1}$.

(1)	(2)	(3)
Identity	$A = (abcd)(efgh)$	$A^2 = (ac)(bd)(eg)(fh)$
$C = (aci)(bgh)(dfe)$	$AC = (agbi)(cfhd)$	$A^2C = (ai)(bf)(dg)(eh)$
$C^2 = (aic)(bhg)(def)$	$AC^2 = (ahfb)(cedi)$	$A^2C^2 = (be)(ci)(dh)(fg)$
$D = (bdi)(agf)(che)$	$AD = (adge)(bhci)$	$A^2D = (ah)(bi)(cg)(ef)$
$D^2 = (bid)(afg)(ceh)$	$AD^2 = (aidf)(begc)$	$A^2D^2 = (ae)(cf)(di)(gh)$
$E = (ige)(cbf)(ahd)$	$AE = (afec)(dhig)$	$A^2E = (ab)(ch)(df)(gi)$
$E^2 = (ieg)(cfb)(adh)$	$AE^2 = (achg)(bfie)$	$A^2E^2 = (af)(bh)(cd)(ei)$
$F = (ifh)(cdg)(aeb)$	$AF = (bdeh)(cgif)$	$A^2F = (ad)(bg)(ce)(fi)$
$F^2 = (ihf)(cgd)(abe)$	$AF^2 = (aeih)(bgfd)$	$A^2F^2 = (ag)(bc)(de)(hi)$

(4)	(5)	(6)
$A^3 = (adcb)(ehgf)$	$B = (ahcf)(bgde)$	$B^3 = (afch)(bedg)$
$A^3C = (afdi)(bcge)$	$BC = (abhi)(cegf)$	$B^3C = (aefi)(bdhc)$
$A^3C^2 = (aegd)(bich)$	$BC^2 = (ageh)(cdfi)$	$B^3C^2 = (adbf)(cghi)$
$A^3D = (aibg)(cdhf)$	$BD = (aedc)(bfgi)$	$B^3D = (bcei)(dfhg)$
$A^3D^2 = (abfh)(cide)$	$BD^2 = (acgb)(idhe)$	$B^3D^2 = (agid)(bhfe)$
$A^3E = (bhed)(cfig)$	$BE = (adig)(befh)$	$B^3E = (acde)(bigf)$
$A^3E^2 = (ahie)(bdfg)$	$BE^2 = (biec)(fdgh)$	$B^3E^2 = (abgc)(dieh)$
$A^3F = (aghc)(beif)$	$BF = (aife)(hdbc)$	$B^3F = (aheg)(cifd)$
$A^3F^2 = (acef)(dgih)$	$BF^2 = (afbd)(hgci)$	$B^3F^2 = (aihb)(cfge)$

(7)	(8)
$AB = (agce)(bfdh)$	$A^3B = (aecg)(bhdf)$
$ABC = (ahgi)(becd)$	$A^3BC = (adei)(chfg)$
$ABC^2 = (abdg)(cfei)$	$A^3BC^2 = (afhe)(bgic)$
$ABD = (afib)(degh)$	$A^3BD = (acfd)(behi)$
$ABD^2 = (bgef)(chid)$	$A^3BD^2 = (ahbc)(dgfi)$
$ABE = (aehf)(bcig)$	$A^3BE = (aigh)(bdce)$
$ABE^2 = (aied)(cgfh)$	$A^3BE^2 = (agdb)(cief)$
$ABF = (acbh)(difg)$	$A^3BF = (abif)(dhge)$
$ABF^2 = (adfc)(bihe)$	$A^3BF^2 = (bfeg)(cdih)$

We seek the conditions under which these 72 substitutions will give rise to a complete set S_{36}^{10} of dihedrons each occurring under two notations (§1) as follows:

$$(14) \qquad j\,a\,b\,a\,\beta\,\gamma\,\delta\,\epsilon\,\rho\,\sigma, \qquad j\,\sigma\,\rho\,\epsilon\,\delta\,\gamma\,\beta\,a\,b\,a,$$

where j is a new (tenth) letter. We may assume that a and b occupy the second and third places, since H_{72}^2 is doubly-transitive. The substitution which interchanges the two sets (14) is

$$(15) \qquad T = (a\sigma)\ (b\rho)\ (a\epsilon)\ (\beta\delta)\ (\gamma).$$

It must belong to H_{72}^2 by hypothesis. It cannot be A^2E, which interchanges a and b; nor A^2D^2 which leaves b fixed; nor A^2C^2 which leaves a fixed. Hence T must be one of the remaining six of period 2:

$$(16) \qquad A^2,\ A^2C,\ A^2D,\ A^2E^2,\ A^2F,\ A^2F^2.$$

We consider here* only the case $T = A^2$. Then

$$\sigma = c,\ \rho = d,\ \gamma = i,\ (a\epsilon)\ (\beta\sigma) = (eg)\ (fh).$$

Now $a \neq e$, since A^2D^2 replaces the triple *abe* by *eba*. Also $\epsilon \neq e$, when $\delta = f$, since A^2E replaces *fedc* by *defh*. We consider in detail the case $a = g$, $\delta = h$. Then the first dihedron (14) is *jabgfihedc*. Applying to it the 36 substitutions† in the blocks (1), (2), (5), (7) of the above table, we obtain the following 36 dihedrons, written in the same order:

j a b g f i h e d c	*j h g d a i c b e f*
j c g h e a b d f i	*j b h f c a i g d e*
j i h b d c g f e a	*j g b e i c a h f d*
j g d f a b e c i h	*j e f i g b h d c a*
j f i a g d c h b e	*j c a b f d e i h g*
j h f e c g d i a b	*j d e a h g b f i c*
j d c i b e a g h f	*j a i h d e f c g b*
j e a c h f i b g d	*j i c g e f d a b h*
j b e d i h f a c g	*j f d c b h g e a i*
j b c h g i e f a d	*j g f c d i b a h e*
j g i b h a d e c f	*j h e i f a g c b d*
j h a g b c f d i e	*j b d a e c h i g f*
j d h e f b c a g i	*j f a h i b d g e c*
j i e c a d h g f b	*j a g e b d i f c h*
j f b d e g i c h a	*j e c b a g f h d i*
j c f a i e g b d h	*j i b f h e c d a g*
j a d i c f b h e g	*j c h d g f a e i b*
j e g f d h a i b c	*j d i g c h e b f a*

This set forms a complete set S^{10}_{36}, as a tedious verification‡ shows.

THE UNIVERSITY OF CHICAGO,
FEBRUARY, 1904.

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* An exhaustive treatment should make use of normalizations by transformations within H^9_{72} so that only independent complete sets shall result. For instance, the 9 substitutions of period 2 are all conjugate within H.

† These together with their products on the left by $T = A$ give the 72 distinct substitutions of H^9_{72}, so that they give all the distinct dihedrons arising.

‡ When so many dihedrons enter as here, the verification may best be made by listing all the forward and backward triples entering. Thus

$$aj\,\{b, c, d, e, f, g, h, i\}'_1 \qquad ab\,\{c, d, e, f, g, h, i, j\}'_1$$

are two of the 81 lines in the list for the present complete set.

ADDENDUM. (AUGUST, 1904).

9. From the conditions (§4) for a set of marks a_i we may derive a more elegant set of equivalent conditions, by means of which the numerical computations for a complete solution are greatly reduced.

Conditions (10), (11), (12), (13) may be combined into the set

$$\begin{vmatrix} a_{i+1} - a_i & a_{j+1} - a_j \\ a_{i+2} - a_i & a_{j+2} - a_j \end{vmatrix} \neq 0 \qquad (i, j = 1, 2, \ldots, p^n - 2;\ i < j),$$

since $a_1 = 0$, $a_2 = 1$. Hence they are the conditions that the functions

$$f_i \equiv (a_{i+2} - a_i)\ (a_{i+1} - a_i) \qquad (i = 1, 2, \cdots, p^n - 2)$$

shall be distinct. Since the a's are distinct, $f_i \neq 0, 1$. Hence conditions (10)–(13) require merely that $f_1, \cdots, f_{p^n-2}$ shall form a permutation of the marks $\neq 0, 1$. Set

$$g_i = f_i - 1, \qquad \pi_i = g_1 g_2 \cdots g_{i-1} g_i.$$

By induction it is readily shown that

$$(a) \qquad a_t = a_{t-1} + \pi_{t-2} \qquad (t = 3, 4, \cdots, p^n).$$

Indeed, the formula is true for $t = 3$ since $a_3 = a_2 + g_1 = f_1$. Next,

$$a_{t+1} = a_{t-1} + f_{t-1}\ (a_t - a_{t-1}) = a_t - \pi_{t-2} + f_{t-1}\,\pi_{t-2} = a_t + \pi_{t-2}\, g_{t-1} = a_t + \pi_{t-1},$$

so that the formula is true for $t + 1$ if true for t. It follows that

$$a_{t+1} = a_{t-1} + \pi_{t-2} f_{t-1}.$$

From the latter it follows by induction from a_j to a_{j+2} that

$$(b) \qquad a_{2k} = 1 + \sum_{i=1}^{k-1} \pi_{2i-1} f_{2i}, \qquad a_{2k+1} = f_1 + \sum_{i=1}^{k-1} \pi^{2i} f_{2i+1}.$$

Finally, we obtain the conditions on the g_i resulting from (9). We note first that the set (9) is equivalent to the set

$$a_{p^n} - a_{p^n-1} = 1,\ a_{p^n-i+1} - a_{p^n-i} = a_{i+1} - a_i \qquad (i = 2, \cdots, q),$$

the first being (9) for $i = 1$, and the second being derived by subtracting (9) for $i = j$ from (9) for $i = j - 1$. In view of (a), these become

$$\pi_{p^n-2} = 1, \quad \pi_{p^n-i-1} = \pi_{i-1} \qquad (i = 2, \cdots, q).$$

Hence we get $$F_i \equiv g_i\, g_{i+1} \cdots g_{p^n-i-1} = 1 \quad (i = 1, 2, \cdots, q).$$

Dividing F_i by F_{i+1}, we get

$$(c) \qquad g_i\, g_{p^n-i-1} = 1 \qquad (i = 1, 2, \cdots, q-1).$$

Also $F_q = 1$ becomes

$$(d) \qquad g_{\frac{1}{2}(p^n-1)} = 1 \quad \text{if } p > 2, \qquad g_{2^{n-1}-1}\, g_{2^{n-1}} = 1 \text{ if } p = 2.$$

Evidently relations (c) and (d) are together equivalent to the set $F_i = 1$.

THEOREM. *Every set of distinct marks* $a_1 = 0$, $a_2 = 1$, $a_i\,(i = 3, \cdots, p^n)$ *which satisfy conditions* (9) – (13) *can be determined as follows: If* $p>2$, *we take in succession as the* g_i $[i = 1, \cdots, \frac{1}{2}(p^n - 3)]$ *each* set of marks for which the* $p^n - 3$ *marks* g_i, $1/g_i$ $[i = 1, \cdots, \frac{1}{2}(p^n - 3)]$ *form a permutation of all the marks* $\neq 0, 1, -1$, *and take* $g_{\frac{1}{2}(p^n-1)} = 1$. *We then compute the* a_i *by* (b), *or more easily by* (a). *We retain only the sets in which the* a_i *are all distinct. If* $p = 2$, *we take as the* g_i $(i = 1, \cdots, 2^{n-1} - 1)$ *each set of marks for which the* $2^n - 2$ *marks* g_i, $1/g_i$ $(i = 1, \cdots, 2^{n-1} - 1$ *form a permutation of all the marks* $\neq 0, 1$.

For $p^n = 11$, the computation was performed in about five hours, including repeated checking. The work was done in tabular form with columns headed

$$g_1, \cdots, g_9,\ \pi_2 = \pi_7 = g_1 g_2,\ \pi_3 = \pi_6 = g_1 g_2 g_3,\ \pi_4 = \pi_5 = g_1 g_2 g_3 g_4,\ a_3, \cdots, a_{11}.$$

We assign to g_1 the values $2, \cdots, 9$ in turn. Taking first $g_1 = 2$, we enter 6 in the column headed g_9 since $2 \cdot 6 \equiv 1 \pmod{11}$. Then g_2 may take any of the values $2, \cdots, 9$ except 2 and 6. For each value of g_1 except 5 and 9, we may exclude a certain value of g_2, different from g_1 and $1/g_1$, and determined so that $a_4 = a_3 + g_1 g_2 = 0$. The complete list of the 20 sets of distinct a_i, and the values of g_1, g_2, g_3, g_4 leading to them, is as follows:

---

* There are $(p^n - 3)(p^n - 5)(p^n - 7) \cdots 4 \cdot 2$ such sets.

g_1	g_2	g_3	g_4	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
2	3	5	8	3	9	6	4	2	10	5	7	8
2	7	5	4	3	6	10	4	9	2	5	7	8
2	9	4	8	3	10	5	9	2	8	4	6	7
3	5	8	2	4	8	7	5	3	2	6	9	10
3	9	6	7	4	9	6	7	8	5	10	2	3
4	6	5	7	5	7	6	10	3	2	4	8	9
4	8	9	6	5	4	6	7	8	10	9	2	3
5	3	8	2	6	10	9	7	5	4	8	2	3
5	6	4	8	6	3	2	5	8	7	4	9	10
6	3	5	8	7	3	5	10	4	6	2	8	9
6	5	8	4	7	4	2	5	8	6	3	9	10
6	7	5	4	7	5	6	10	3	4	2	8	9
7	3	5	2	8	7	2	3	4	10	9	5	6
7	9	4	6	8	5	4	9	3	2	10	6	7
8	2	5	3	9	3	6	4	2	5	10	7	8
8	9	4	6	9	4	6	7	8	10	5	2	3
9	4	6	7	10	2	9	3	8	4	7	5	6
9	6	3	7	10	9	6	7	8	5	4	2	3
9	6	7	3	10	9	2	3	4	8	7	5	6
9	7	3	2	10	7	9	2	6	8	5	3	4

As a further check, it was verified that each of the resulting 20 sets a_i satisfies conditions (9). Each set leads to a complete set S_{56}^{12}.

ON FUNCTIONS DEFINED BY AN INFINITE SERIES OF ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

By M. B. Porter

In the Annals of Mathematics, ser. 2, vol. 3 (1901–02), p. 25, Professor Osgood has proved among others the following theorem.*

If the infinite series

$$f_1(z) + f_2(z) + \cdots$$

whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z pertaining to a set of points which is everywhere dense throughout T (and which, in particular, may be enumerable), and if furthermore, the relation

$$|f_1(z) + f_2(z) + \cdots + f_n(z)| \leqq G$$

holds for all points z of this set (and hence of T), and for all values of n, G being a positive constant, then the series converges for all values of z in T and the function $F(z)$ defined by the series:

$$F(z) = f_1(z) + f_2(z) + \cdots$$

is analytic throughout T.

The purpose of this note is to prove by the elementary methods of the Cauchy theory the following generalization of this theorem.

If the infinite series

$$f_1(z) + f_2(z) + \cdots$$

whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z pertaining to a point set (a) whose limiting points are everywhere dense on a closed rectifiable contour C lying inside of T, and if the relation

$$|f_1(z) + f_2(z) + \cdots + f_n(z)| < G$$

* For another proof of this theorem see a paper by Arzelà in the Annals of Mathematics, ser. 2, vol. 5, p. 51 (1904).

holds for all points z of T and for all values of n, G being a positive constant, then the series converges for all values of z in T and defines an analytic function of z in T.

1. We shall first prove that

$$S(z) = \lim_{n=\infty} S_n(z) = \lim_{n=\infty} [f_1(z) + f_2(z) + \cdots + f_n(z)]$$

converges uniformly for all the points of the contour C. Denote by $\rho \neq 0$ the lower limit of the distances of the points (a) to the boundary of T. We have then,

$$S_n(z) = S_n(a) + b_1(z-a) + b_2(z-a)^2 + \cdots, \qquad |z-a| < \rho,$$

where a denotes any one of the points of the set (a), and where by Cauchy's inequality:

$$|b_n| \leqq \frac{G}{\rho^n}.$$

So that

$$|S_n(z) - S_n(a)| < G \sum_1^\infty \frac{|z-a|^n}{\rho^n} < M|z-a|, \quad \begin{cases} n = 1, 2, 3, \cdots, \\ |z-a| < \rho' < \rho, \end{cases}$$

where M is fixed for all the set (a).
Further

$$[1] \qquad |S_{n+p}(z) - S_n(z)| < |S_{n+p}(a) - S_n(a)| + 2M|z-a|, \qquad p = 1, 2, 3, \cdots.$$

If now z be a limiting point of (a), a can be so chosen that $2M|z-a| < \frac{\epsilon}{2}$.
Let a be so chosen and held fixed, then since $\lim_{n=\infty} S_n(a)$ exists; an N can be determined so that

$$|S_{n+p}(a) - S_n(a)| < \frac{\epsilon}{2}, \qquad n > N,\ p = 1, 2, 3, \cdots.$$

Thus

$$|S_{n+p}(z) - S_n(z)| < \epsilon, \qquad M > N,\ p = 1, 2, 3, \cdots,$$

which shows that $\lim_{n=\infty} S_n(z) = S(z)$ exists (z as above stated denoting a point of C).

That $S(z)$ converges uniformly to its limit is shown thus; let $\bar{z}$ denote a point of C different from z such that

$$|z - \bar{z}| < \rho' < \rho;$$

then precisely as in (1) we show that

$$|S_{n+p}(z) - S_{n+p}(z)| < |S_{n+p}(\bar{z}) - S_n(\bar{z})| + 2M|z - \bar{z}|, \qquad p = 1, 2, 3, \cdots.$$

If now we choose $\bar{z}$ so that

$$2M|z - \bar{z}| < \frac{\epsilon}{2},$$

we get

$$|S_{n+p}(\bar{z}) - S_n(\bar{z})| < \epsilon, \qquad n > N, \quad p = 1, 2, 3, \cdots,$$

provided

$$|z - \bar{z}| < \frac{\epsilon}{4M},$$

which proves that $S(z)$ converges uniformly on C.

2. We shall now prove that $S(z)$ *converges and converges uniformly in any region lying inside of* C *and therefore by a well known theorem represents an analytic function in such region.*

To do this we shall need the following lemma, to which, though doubtless well known, I can give no reference in the literature.

LEMMA. If a series of functions

$$f_1(z) + f_2(z) + \cdots,$$

single valued and analytic inside a region S and continuous along the boundary C of S, supposed rectifiable, converges uniformly along C, then

$$f_1(z) + f_2(z) + \cdots$$

converges uniformly and represents an analytic function of z in any region lying inside of C.

Proof. Let t denote the affix of any point on C, L any region inside of C such that the minimum distance of the points of L to the contour C is $\sigma \neq 0$, and z any point of L; then by Cauchy's integral formula we have:

$$S_n(z) = \frac{1}{2\pi i}\int_C \frac{S_n(t)dt}{t - z},$$

where

$$S_n(z) = \sum_1^n f_i(z);$$

also $$|S_{n+p}(z) - S_n(z)| = \frac{1}{2\pi}\left|\int_C \frac{S_{n+p}(t) - S_n(t)}{t-z}\,dt\right|$$
$$< \frac{1}{2\pi}\frac{\epsilon}{\sigma}C < \epsilon',$$

where C is the length of the contour C, and where, since $S(t) = \lim_{n=\infty} S_n(t)$ is uniformly convergent, a number N can be found such that

$$|S_{n+p}(t) - S_n(t)| < \epsilon, \qquad n > N.$$

Thus ϵ' can be made as small as we please for all points of L at once by a suitable choice of N, which demonstrates the Lemma.

The application of this lemma shows that the theorem stated at the beginning of §2 is true, and applying a well known theorem of Stieltje's,* $S(z)$ is seen to be analytic throughout the region T.

An examination of the foregoing proof shows that it is valid if the condition

$$|\sum_1^n f_i(z)| < G$$

is merely imposed for the part of T lying outside of a region of the type styled L in §2.

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* See *Ann. de la Faculté des Sciences de Toulouse*, vol. 8 (1894), p. J. 56, or Professor Osgood's paper above cited.

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